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Master's thesis

Singular weight products on lattices with small discriminant

Fabian Kertels

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Supervisor: Prof. Dr. Nils Scheithauer

Second referee: Prof. Dr. Jan Hendrik Bruinier

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Introduction

This thesis aims at finding, with certain restrictions, lattices that carry a so-called holomorphic automorphic product of singular weight. More precisely, an algorithm is developed that checks, up to an arbitrary finite bound on the cardinality of the discriminant form, all lattices of prime level for the existence of such an automorphic product. In a basic version, which is fully worked out here and applied to a number of examples, it is still limited to levels 2 and 3.

Before it comes to the computational part, the text tries to lead to the theory of modular forms for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ in an introduction which is as detailed as possible within the given scope. These objects are a generalization of elliptic modular forms. They take values in the group algebra of the discriminant form associated with a lattice and their transformation behaviour involves a particular representation of $\mathrm{SL}_2(\mathbb{Z})$ on this complex vector space. The *singular theta correspondence* found by Borcherds [Bor98] maps such modular forms to automorphic forms for orthogonal groups, i.e. to meromorphic functions on a complex manifold that satisfy certain transformation conditions regarding scalar multiplication and lattice automorphisms acting on the manifold.

The correspondence can be described as follows. Let L be an even lattice of type $(n, 2)$ with an even integer $n > 2$, $D = L'/L$ its discriminant form, θ the Siegel theta function of L and F a modular form for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ of weight $(2-n)/2$. Assume that all Fourier coefficients in the principal part of F are integral and the constant coefficient of the 0-component is even. Then on some Grassmannian manifold,

$$\Psi(F)(Z) = \exp\left(\int_{\mathcal{F}} \big(F(x+iy), \overline{\theta(Z, x+iy)}\big) y \frac{\mathrm{d}x\mathrm{d}y}{y^2}\right)$$

defines an automorphic form for the discriminant kernel of $O(L)^+$. Here $(\cdot\,,\cdot)$ denotes the Hermitian inner product of $\mathbb{C}[D]$, $\mathcal{F}$ is the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ on the complex upper half plane and the value of the integral is defined by means of regularization. The function $\Psi(F)$ can be shown to have interesting product expansions around certain points and is therefore called the *automorphic product* associated with L and F. Its weight and divisor are determined by the constant coefficient of the 0-component and by the principal part of F, respectively.

One can prove that there is a general lower bound on the weight of holomorphic automorphic *forms* for the discriminant kernel of $O(L)^+$ that only depends on n. However, holomorphic automorphic products of this *singular weight*, which is given by $(n-2)/2$, seem to be very rare. Up to now, all known examples are denominator functions of infinite-dimensional Lie superalgebras. This text shall take another step towards the classification of these holomorphic automorphic products of singular weight. The method was proposed in a recent article by Scheithauer [Sch14] on which the thesis relies heavily. The main result of its computational application is the following.

Let L be an even lattice of level $p \in \{2, 3\}$ and genus $II_{n,2}(D)$, $n > 2$ even, such that $|D| < 100$ and L splits a hyperbolic plane. Suppose L carries a holomorphic automorphic product of singular weight. Then the genus of L is one of the following:

$$II_{10,2}(2_{II}^{+2}), II_{10,2}(2_{II}^{+4}), II_{10,2}(2_{II}^{+6})$$

and

$$II_{26,2}(2_{II}^{+2}), II_{26,2}(2_{II}^{+4}), II_{26,2}(2_{II}^{+6})$$

if $p = 2$,

$$II_{8,2}(3^{-3})$$

and

$$II_{26,2}(3^{-2}), II_{26,2}(3^{+4})$$

if $p = 3$. Conversely, every lattice whose genus is contained in this list carries a holomorphic automorphic product of singular weight.

In particular, for said discriminant forms there are no other lattices carrying holomorphic automorphic products of singular weight than those that have already been constructed elsewhere.

The algorithm that yields this result takes the following main steps. For fixed level p and rank n_p, up to isomorphy there are infinitely many discriminant forms of type $(n, 2)$, $n > 2$ even, and cardinality p^{n_p}, at most one for each n. Under the assumption that L carries a holomorphic automorphic product of singular weight, there is an upper bound on the signature that only depends on p and n_p and can easily be computed. This leaves us with finitely many possible n. For given n another criterion can be extracted from the proof of the bound, which allows us to discard some more lattices. For those remaining it yields an upper bound on the pole orders of the components of a hypothetical vector valued modular form F that could induce a holomorphic automorphic product of singular weight. This last step requires that the lattice split a hyperbolic plane.

Now for fixed n the principal part of F can be expressed as a tuple of complex numbers representing the potentially non-zero Fourier coefficients. The different conditions that F must satisfy are equivalent to certain conditions on its principal part. First of all, the coefficients have to be integral for the singular theta correspondence to apply. Next, the holomorphy condition leads to a system of linear inequalities which, roughly speaking, mean that the principal part has to be almost nonnegative. The weight of the holomorphic automorphic product depends, as already mentioned, only on one of the constant coefficients and therefore not directly on the principal part. Luckily, knowledge about a vector valued Eisenstein series allows to compute this coefficient from the principal part via the bilinear *pairing* operation. The condition then translates to a linear equation in the coefficients. Finally, the computationally most complex part is the so-called obstruction theory. Not for every choice of coefficients there is a modular form for the Weil representation which has the corresponding principal part and the desired weight. However, this is true if and only if the constant coefficient of the pairing of the principal part and any element of a certain space of cusp forms vanishes. A system of generators for this space can be constructed from classical scalar valued cusp forms. Then the obstruction theory manifests as a homogeneous system of linear equations.

In total, an integer linear program has been generated whose feasibility is equivalent to the existence of the function F. This can be solved now, which in practice becomes a real nuisance when demanding exact solutions, to obtain the results stated above. Along the way we can collect additional data on the different parts of the problem which hopefully might give rise to new conjectures on the classification issue.

The rest of the text is organized as follows.

Chapter 1 summarizes some facts about lattices and discriminant forms. Most proofs are, of course, omitted, but to understand the results no more than basic algebra knowledge should be needed.

Chapter 2 gives a condensed introduction to scalar valued modular forms with an emphasis on those with character and the construction of eta products.

In Chapter 3, based on the work done in the previous two chapters, the theory of modular forms for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ is described.

Afterwards the concept of automorphic forms for orthogonal groups and the singular theta correspondence are briefly sketched.

Chapter 5 contains the computational results and a more detailed description of the approach that was used. Some final remarks follow.

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Chapter 1

Lattices and discriminant forms

Although lattices over the integers are rather specific objects, they have been widely studied in different branches of mathematics. As $\mathbb{Z}$-modules endowed with some quadratic form and usually considered as subsets of a finite-dimensional rational or real vector space, lattices not only play important roles where algebra resorts to analysis, such as elliptic curve theory, modular forms and analytic number theory in general. They are also of interest in their own right and closely related to sphere packing and tiling problems.

The following summary of basic knowledge on this topic resembles those given in other theses, such as [Sch13] or [Dit13]. It first introduces the more general concept of a quadratic module and then proceeds to lattices and discriminant forms. The latter arise by taking certain quotients of lattices. They partly form the basis of the theory of vector valued modular forms in Chapter 3, whereas lattices themselves can also be used to construct scalar valued modular forms, as done in Chapter 2.

1.1 Quadratic modules

Throughout this section, which follows [BGHZ08, Part 2, Section 2.1], let R be a commutative ring with unity and M a finitely generated R-module.

Definition 1.1. A *quadratic form* on M is a map $Q : M \to R$ such that $Q(rx) = r^2Q(x)$ holds for all $r \in R$, $x \in M$ and

$$B : M \times M \to R : (x, y) \mapsto Q(x + y) - Q(x) - Q(y)$$

is a bilinear form. In that case we call the pair (M, Q) – or simply the module M without mentioning the quadratic form – a *quadratic module over* R (or *quadratic space* if R is a field).

Now let Q be a quadratic form on M and B the corresponding bilinear form. Note that B is necessarily symmetric. Furthermore, if 2 is invertible in R, we can regain Q from B via $Q(x) = \frac{1}{2}B(x, x)$. The value of Q at $x \in M$ is also referred to as the *norm of* x (even though Q does not quite behave like a norm in the sense of analysis). Elements of norm zero are called *isotropic*, the rest being *anisotropic*. Two elements $x, y \in M$ are *orthogonal* if $B(x, y) = 0$. For any subset A of M, the *orthogonal complement* of A is given by

$$A^{\perp} = \{x \in M : B(x, y) = 0 \text{ for all } y \in M\} .$$

This is always a submodule of M, no matter of which form A is. Further, one easily verifies that $A \subseteq (A^\perp)^\perp$ and $A^\perp \subseteq B^\perp$ for $B \subseteq A$, which in turn yield $A^\perp = \big((A^\perp)^\perp\big)^\perp$. The quadratic module (M, Q) and its quadratic form Q are called *degenerate* if the bilinear form B is, i.e. if there is some non-zero $x \in M$ such that $B(x, y) = 0$ for all $y \in M$. Hence M is *non-degenerate* exactly if $M^\perp = \{0\}$.

Definition 1.2. Suppose M is free with basis $(e_1, \dots, e_m)$. The symmetric matrix $G = (B(e_i, e_j))_{1 \le i,j \le m}$ is called the *Gram matrix of M with respect to the basis* $(e_1, \dots, e_m)$. Its determinant is independent of the basis up to multiplication by an element in $(R^\times)^2$ and also called the *discriminant* of M, denoted by $\operatorname{disc}(M)$.

We get the familiar relation

$$B\Big(\sum_{i=1}^{m} \lambda_i e_i, \sum_{j=1}^{m} \mu_j e_j\Big) = (\lambda_1, \dots, \lambda_m) \cdot G \cdot (\mu_1, \dots, \mu_m)^\mathsf{T}$$

for all $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_m \in R$. In the following sections, the Gram matrix will also provide a formula for the cardinality of a discriminant form.

We proceed to mappings between quadratic modules.

Definition 1.3. Let (M, Q), (M_0, Q_0) be quadratic modules over R. An R-linear map $\sigma : M \to M_0$ is called *isometry* if it is injective and satisfies $Q_0(\sigma(x)) = Q(x)$ for all $x \in M$. Two quadratic modules are *isometric* if there is a bijective isometry between them. The set of all isometries of M onto itself, the *orthogonal group* of M, will be denoted by $O(M)$. It forms a subgroup of the group $\operatorname{Aut}(M)$ of module automorphisms of M.

If (M, Q) is a quadratic module over R and M_0 another finitely generated R-module, Q can be extended with zero to the direct sum $M \oplus M_0$, i.e. $Q\big((x, y)\big) := Q(x)$ for all $x \in M$, $y \in M_0$. Just as easily one checks that sums of quadratic forms are quadratic forms again. Thus the following is a reasonable definition.

Definition 1.4. Let (M, Q), (M_0, Q_0) be quadratic modules over R. Their *orthogonal direct sum* is defined as the module $M \oplus M_0$ with the quadratic form

$$Q \oplus Q_0 : M \oplus M_0 \to R : (x, y) \mapsto Q(x) + Q_0(y).$$

With the usual embeddings this definition yields $M \subseteq (M_0)^\perp$ and $M_0 \subseteq M^\perp$.

1.2 Lattices

Now we restrict ourselves to the cases $R = \mathbb{Q}$ and $R = \mathbb{Z}$.

Definition 1.5. Let (V, Q) be a finite-dimensional non-degenerate quadratic space over $\mathbb{Q}$. A free $\mathbb{Z}$-submodule $L \subseteq V$ (together with the quadratic form induced by Q) is a *lattice* in V if $V = L \otimes_\mathbb{Z} \mathbb{Q}$.

Equivalently, a lattice can be defined as a discrete subgroup of V (with respect to the Euclidean topology) that generates V as a $\mathbb{Q}$-vector space, or as a subset of the form $\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_m$, where $(e_1, \dots, e_m)$ is a $\mathbb{Q}$-basis of V. For every lattice $L \subseteq V$ and every $a \in \mathbb{Z} \setminus \{0\}$, the lattice obtained by *rescaling* L with a is denoted by $L(a)$. It is the same module, but endowed with the quadratic form $a \cdot Q$ instead of Q.

Since the quadratic form Q can be described by a rational symmetric matrix, Sylvester's law of inertia from linear algebra applies to it. This justifies the following definition.

Definition 1.6. Let (V, Q) be a finite-dimensional non-degenerate quadratic space over $\mathbb{Q}$ and L a lattice in V. Let b^+ and b^- denote the numbers of positive and negative squares in Q, respectively, i.e. the matrix of Q has b^+ positive and b^- negative (real) eigenvalues. Then we call the pair (b^+, b^-) the *type* of both V and L and $\operatorname{sign}(V) = \operatorname{sign}(L) = b^+ - b^-$ their *signature*.

Obviously types and signatures add up when orthogonal direct sums are formed. A lattice L in (V, Q) is called *integral* if the bilinear form B associated with Q takes only integral values on L, i.e. $B(x, y) \in \mathbb{Z}$ for all $x, y \in L$. It is *even* if also Q itself is integral valued on L or, equivalently, $B(x, x)$ is even for all $x \in L$. Hence every even lattice is a quadratic module over $\mathbb{Z}$. Conversely, every free non-degenerate quadratic $\mathbb{Z}$-module L of finite rank can be viewed as an even lattice in the ambient space $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$ since the quadratic form of L uniquely extends to V. In this case the orthogonal groups of L as a quadratic module and as a lattice coincide, where the latter in general is defined by $O(L) := \{\sigma \in O(V) : \sigma(L) = L\}$. We will often drop the ambient space when working with even lattices.

Definition 1.7. Let (V, Q) be a finite-dimensional non-degenerate quadratic space over $\mathbb{Q}$ with associated bilinear form B and L a lattice in V. The set

$$L' = \{x \in V : B(x, y) \in \mathbb{Z} \text{ for all } y \in L\}$$

is called the *dual lattice* of L.

Indeed, L' is a lattice again: Let $(e_1, \dots, e_m)$ be a basis of V such that $L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_m$. The space is non-degenerate, whence its Gram matrix G with respect to the basis $(e_1, \dots, e_m)$ is invertible. With $G^{-1} = (s'_{ij})_{1 \le i,j \le m}$ define $e'_i := \sum_{j=1}^m s'_{ij} e_j$ for all $i \in \{1, \dots, m\}$. Then for all i, j we have $B(e'_i, e_j) = \delta_{ij}$ (the j-th entry of $(s'_{i1}, \dots, s'_{im}) \cdot G$), i.e. $(e'_1, \dots, e'_m)$ is the dual basis to $(e_1, \dots, e_m)$. Now it is easy to see that $L' = \mathbb{Z}e'_1 \oplus \cdots \oplus \mathbb{Z}e'_m$.

This construction implies that G^{-1} is the base change matrix from $(e_1, \dots, e_m)$ to $(e'_1, \dots, e'_m)$. Hence $(L')' = L$ and the Gram matrix of L' with respect to the basis $(e'_1, \dots, e'_m)$ is given by $(G^{-1})^\intercal G G^{-1} = G^{-1}$, too. Note that it also makes perfect sense to assign the name *dual* to the structure L': for each $x \in L'$ the map $B(x, \cdot)$ is $\mathbb{Z}$-linear on L with values in $\mathbb{Z}$. This correspondence is not only injective because of the non-degeneracy of (V, Q), but also surjective, as was just shown implicitly by constructing the dual basis $(e'_1, \dots, e'_m)$. Thus we have

$$L' = \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) = \{x' \in V' : x'(x) \in \mathbb{Z} \text{ for all } x \in L\}$$

where $V' = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ is the usual dual vector space to V. A lattice L is integral if and only if $L \subseteq L'$. If equality holds, it is called *unimodular*. Obviously, if $M \subseteq L$ is another lattice – a *sublattice* of L –, we have $L' \subseteq M'$.

Definition 1.8. Let (V, Q) be a finite-dimensional non-degenerate quadratic space over $\mathbb{Q}$ with associated bilinear form B and L a lattice in V. The *level* of L is the smallest positive integer N such that $NQ(x) \in \mathbb{Z}$ for all $x \in L'$. If L is even, for any particular non-zero $x \in L$ the *level of* x is the natural number N_x defined by $B(x, L) = N_x\mathbb{Z}$. If $\mathbb{Q}x \cap L = \mathbb{Z}x$, x is called *primitive*.

Proposition 1.9. *Let L be a lattice of level N. Then $NL' \subseteq L$.*

Proof. Choose a $\mathbb{Z}$-basis $(e_1, \ldots, e_m)$ for L and as usual denote by B the bilinear form, by G the corresponding Gram matrix and by $(e'_1, \ldots, e'_m)$ the dual basis. As was proved above, $G^{-1} = \big(B(e'_i, e'_j)\big)_{1\le i,j\le m}$, so N can alternatively be characterized as the smallest natural number such that NG^{-1} has integral entries and those on the diagonal are even. Since at the same time G^{-1} is the base change matrix from $(e_1, \ldots, e_m)$ to $(e'_1, \ldots, e'_m)$, this shows that $Ne'_1, \ldots, Ne'_m$ are integral linear combinations of $e_1, \ldots, e_m$ and hence $NL' = \mathbb{Z}Ne'_1 \oplus \cdots \oplus \mathbb{Z}Ne'_m \subseteq L$. $\square$

Let us consider some examples.

Example 1.10. Equip $\mathbb{Q}^2$ with the quadratic form $Q : \mathbb{Q}^2 \to \mathbb{Q} : (x_1, x_2) \mapsto x_1x_2$. With respect to the canonical basis it has the Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, whose eigenvalues are -1 and 1. Hence $(\mathbb{Z}^2, Q)$ is an even unimodular lattice of type $(1, 1)$. Every lattice isometric to this one is called *hyperbolic plane*. It is of great importance since it appears in many occasions and has some handy properties, such as existence of primitive elements of arbitrary integral norm. Therefore many theorems in the later parts of this thesis will require a lattice to split a hyperbolic plane, i.e. to have an orthogonal sum decomposition containing a hyperbolic plane.

Example 1.11. Now consider the quadratic form $Q : \mathbb{Q}^2 \to \mathbb{Q} : (x_1, x_2) \mapsto x_1^2 - x_1x_2 + x_2^2$ instead. The even lattice $(\mathbb{Z}^2, Q)$ is commonly denoted by A_2. Its Gram matrix with respect to the canonical basis is $G = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. Its eigenvalues are 1 and 3, so A_2 is of type $(2, 0)$ or, in other words, positive-definite. Inverting the Gram matrix yields $A'_2 = \mathbb{Z}(\frac{2}{3}, \frac{1}{3})^\intercal \oplus \mathbb{Z}(\frac{1}{3}, \frac{2}{3})^\intercal$, which further shows us that A_2 has level 3.

Example 1.12. As a last example, take the set $\big\{(x_1, x_2, x_3, x_4)^\intercal \in \mathbb{Q}^4 : x_1 + x_2 + x_3 + x_4 \in 2\mathbb{Z}\big\}$ together with the quadratic form $\mathbb{Q}^4 \to \mathbb{Q} : (x_1, x_2, x_3, x_4)^\intercal \mapsto \frac{1}{2}(x_1^2+x_2^2+x_3^2+x_4^2)$ (i.e. the bilinear form is the standard scalar product). Obviously it is a $\mathbb{Z}$-submodule of the non-degenerate quadratic module $\mathbb{Q}^4$. Since a basis is given by $\big((1, 1, 0, 0)^\intercal, (1, -1, 0, 0)^\intercal, (0, 1, -1, 0)^\intercal, (0, 0, 1, -1)^\intercal\big)$, it is indeed an even lattice. It has type $(4, 0)$ and is usually referred to as D_4. The Gram matrix with respect to the above basis and its inverse are

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 1 & \frac{1}{2} \\ -1 & 1 & 2 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 1 & 1 \end{pmatrix},$$

respectively. So D_4 has level 2 (cf. the proof of Proposition 1.9).

1.3 Discriminant forms

Let L be any lattice. Since $(\mathbb{Z}^\times)^2 = 1$, the discriminant $\operatorname{disc}(L)$ of L is truly unique as an element of the integers (cf. Definition 1.2). Now assume L to be integral. Then $L \subseteq L'$ and we can consider the quotient L'/L. The elementary divisors theorem in particular shows that this abelian group is finite:

Theorem 1.13. *Let L be an integral lattice. Then $|L'/L| = |\operatorname{disc}(L)|$.*

Proof. By virtue of the elementary divisors theorem, there are a basis $(e_1, \dots, e_m)$ of the $\mathbb{Z}$-module L' and positive integers $a_1, \dots, a_m$ such that $(a_1e_1, \dots, a_me_m)$ is a $\mathbb{Z}$-basis of L. This yields the usual group isomorphism

$$L'/L \to \bigoplus_{i=1}^m \mathbb{Z}/a_i\mathbb{Z} : \sum_{i=1}^m \lambda_i e_i + L \mapsto (\lambda_1 + a_1\mathbb{Z}, \dots, \lambda_m + a_m\mathbb{Z}) .$$

Thus $|L'/L| = \det(D)$, where D is the diagonal matrix with entries $a_1, \dots, a_m$ on the diagonal. Now let $G = \big(B(a_ie_i, a_je_j)\big)_{1\le i,j\le m}$ and $G' = \big(B(e_i, e_j)\big)_{1\le i,j\le m}$ be Gram matrices for L and L', respectively. Then by definition $G = DG'D$ and therefore

$$\operatorname{disc}(L) = \det(G) = \det(D)^2 \det(G') = \det(D)^2 \operatorname{disc}(L') = \det(D)^2 \det(G^{-1}).$$

Here we have used that G^{-1} is another Gram matrix for L'. This finally shows that $\det(D) = |\det(G)| = |\operatorname{disc}(L)|$, thereby proving the assertion. □

From now on, we also presume that L is an even lattice. Then for all $x \in L'$, $y \in L$ we have $Q(x + y) - Q(x) = B(x, y) + Q(y) \in \mathbb{Z}$. Thus the maps

$$q : L'/L \to \mathbb{Q}/\mathbb{Z} : x + L \mapsto Q(x) + \mathbb{Z}$$

and

$$b : L'/L \times L'/L \to \mathbb{Q}/\mathbb{Z} : (x + L, y + L) \mapsto B(x, y) + \mathbb{Z}$$

are well-defined. Now abstracting from the original lattice leads to the notion of a discriminant form.

Definition 1.14. Let D be a finite abelian group. A map $q : D \to \mathbb{Q}/\mathbb{Z}$ is called *finite quadratic form* if $q(a\gamma) = a^2q(\gamma)$ for all $a \in \mathbb{Z}$, $\gamma \in D$ and

$$b : D \times D \to \mathbb{Q}/\mathbb{Z} : (\gamma, \beta) \mapsto q(\gamma + \beta) - q(\gamma) - q(\beta)$$

is $\mathbb{Z}$-bilinear. If b is non-degenerate, the pair (D, q) is a *discriminant form*. The *level* of a discriminant form (D, q) is the smallest positive integer N such that $Nq(\gamma) = 0 \bmod 1$ for all $\gamma \in D$.

Obviously the properties making Q into a quadratic form on L carry over analogously to the map q on L'/L. For the non-degeneracy note that if $B(x, y) \in \mathbb{Z}$ for all $y \in L'$ and some $x = \lambda_1e_1 + \cdots + \lambda_me_m$, where $(e_1, \dots, e_m)$ is a basis of L, $(e_1', \dots, e_m')$ the corresponding dual basis and $\lambda_1, \dots, \lambda_m \in \mathbb{Q}$, we already have

$\lambda_i = B(x, e_i') \in \mathbb{Z}$ for all $i \in \{1, \ldots, m\}$, that is, $x \in L$. Thus L'/L is a discriminant form of the same level as L, called *the discriminant form of* L. It is trivial if and only if L is unimodular. Moreover, we will see below that conversely every discriminant form arises from an even lattice by means of this construction.

If (D, q) is a discriminant form and $\beta, \gamma \in D$, we will mostly write $\gamma\beta$ for the value $b(\gamma, \beta)$ of the finite bilinear form and consistently $\gamma^2/2$ instead of $q(\gamma)$. Of course 2 is by no means invertible in $\mathbb{Q}/\mathbb{Z}$, so this has to be considered a mere notation. Like for lattices, we will often just write D for the discriminant form (D, q). A homomorphism of discriminant forms is naturally defined as a group homomorphism preserving the values of the quadratic form, which also implies a notion of isomorphy. Orthogonality and direct sums are defined exactly as in the case of lattices, too: two elements $\beta, \gamma \in D$ are *orthogonal* if $\gamma\beta = 0 \bmod 1$, the *orthogonal complement* of a subset $A \subseteq D$ is the subgroup $\{\gamma \in D : \gamma\beta = 0 \bmod 1 \text{ for all } \beta \in A\}$ and the *orthogonal direct sum* of two discriminant forms is their direct sum together with the sum of the finite quadratic forms on the individual components.

If A is a subgroup, the homomorphisms $A \to \operatorname{Hom}(D/A^\perp, \mathbb{Q}/\mathbb{Z}) : \gamma \mapsto (\gamma, \cdot)$ and $D/A^\perp \to \operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}) : \gamma + A^\perp \mapsto (\gamma, \cdot)$ are injective, so that

$$|A| \leq |\operatorname{Hom}(D/A^\perp, \mathbb{Q}/\mathbb{Z})| = |D/A^\perp| \leq |\operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z})| = |A|.$$

We successively deduce that these are isomorphisms, that $|D| = |A||A^\perp|$ and $(A^\perp)^\perp = A$. If B is another subgroup, we further get $(A + B)^\perp = A^\perp \cap B^\perp$ and $(A \cap B)^\perp = A^\perp + B^\perp$. Also note that orthogonal summation and formation of discriminant forms from lattices commute, i.e.

$$(L \oplus M)'/(L \oplus M) = (L' \oplus M')/(L \oplus M) = L'/L \oplus M'/M$$

for even lattices L, M.

Let us briefly return to the examples given in the previous section.

Example 1.15. If L is the hyperbolic plane from Example 1.10, L'/L is the trivial discriminant form since L is unimodular. More generally, for a rescaled hyperbolic plane $L(a)$, $a \in \mathbb{Z} \setminus \{0\}$, we can easily compute the dual lattice directly and get $a^{-1}\mathbb{Z} \oplus a^{-1}\mathbb{Z}$. Therefore $L(a)'/L(a)$ has level a and, as a group, is isomorphic to $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/a\mathbb{Z}$. In the other cases, Theorem 1.13 shows $|A_2'/A_2| = |\operatorname{disc}(A_2)| = |\det\left(\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)\right)| = 3$ and $|D_4'/D_4| = |\operatorname{disc}(D_4)| = 4$, hence $A_2'/A_2 = \mathbb{Z}/3\mathbb{Z}$ and $D_4'/D_4 = \mathbb{Z}/4\mathbb{Z}$ or $D_4'/D_4 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Proposition 1.9 we have $2D_4' \subseteq D_4$, so the latter is the case.

The following existence statement, which has already been announced above, can be found as part of Theorem 1.3.2 in [Nik80]:

Theorem 1.16. *Let (D, q) be a discriminant form. Then there is an even lattice (L, Q) such that L'/L and D are isomorphic as discriminant forms.*

Nikulin's article also proves and gathers many other important results concerning discriminant forms. For instance, in view of the previous theorem it is natural to ask whether there is any kind of uniqueness.

Theorem 1.17. [Nik80, Theorem 1.3.1] *Two even lattices L, L_0 have isomorphic discriminant forms if and only if there are even unimodular lattices U, U_0 such that $L \oplus U$ and $L_0 \oplus U_0$ are isometric.*

In order to use Theorem 1.16 to reasonably transfer the definition of the signature to discriminant forms, we additionally need the following:

Theorem 1.18. [Nik80, Theorem 1.1.1] *An even unimodular lattice of type (b^+, b^-) with $b^+, b^- \geq 0$ exists if and only if $b^+ - b^-$ is divisible by 8.*

Definition 1.19. Let D be a discriminant form. Then the *signature* of D is defined as $\operatorname{sign}(D) = \operatorname{sign}(L) + 8\mathbb{Z}$, where L is an even lattice such that $D = L'/L$.

For some purposes it is convenient to classify even lattices by an equivalence relation which is, on the one hand, somewhat weaker than isometry. On the other hand, one still wants to distinguish between lattices of different type. Although we have just seen that a discriminant form D imposes certain limitations on any lattice L satisfying $L'/L = D$, Theorems 1.17 and 1.18 show that these are not even sufficient to determine the signature of L completely. So isomorphy of discriminant forms is not enough. A suitable notion is that of *genus*. Its original definition involves so-called equivalence over the p-adic numbers and has also a topological meaning. However, it turns out that the following definition is equivalent (cf. [CS98, Section 15.7] and [Nik80, Corollary 1.9.4]).

Definition 1.20. Let L be an even lattice of type (b^+, b^-) with discriminant form D. The *genus* of L is the set of all even lattices of type (b^+, b^-) whose discriminant form is isomorphic to D. It is denoted by $II_{b^+,b^-}(D)$ or, if L is unimodular, simply by II_{b^+,b^-}.

Up to now, we only know that for every discriminant form D there is a non-empty genus $II_{b^+,b^-}(D)$. Then b^+, b^- have to satisfy some obvious conditions. We might therefore ask which additional criteria, if any, a genus has to fulfil in order to be non-empty or, conversely, to contain only one lattice (up to isometry). As for the first question, in [Nik80, Theorem 1.10.1] conditions can be found that are even equivalent to $II_{b^+,b^-}(D)$ being non-empty. The following separate necessary and sufficient conditions, which are consequences of said theorem and less complicated to use, still apply to most cases. Let $l(G)$ denote the minimum number of generators of a group G.

Theorem 1.21. *If a genus $II_{b^+,b^-}(D)$ is non-empty, the integers b^+, b^- and the discriminant form D must satisfy $b^+ \geq 0$, $b^- \geq 0$, $b^+ - b^- = \operatorname{sign}(D) \bmod 8$ and $b^+ + b^- \geq l(D)$.*

Theorem 1.22. *Let D be a discriminant form and $b^+, b^- \in \mathbb{Z}$. If simultaneously $b^+ \geq 0$, $b^- \geq 0$, $b^+ - b^- = \operatorname{sign}(D) \bmod 8$ and $b^+ + b^- > l(D)$, there is an even lattice of type (b^+, b^-) having discriminant form D.*

As for the uniqueness question, once again Nikulin provides a sufficient condition whose weaker and more easy-to-use corollary shall be cited here.

Theorem 1.23. [Nik80, Corollary 1.13.3] *Let D be a discriminant form and b^+, $b^- \in \mathbb{Z}$. If $b^+ \geq 1$, $b^- \geq 1$ and $b^+ + b^- \geq 2 + l(D)$, all even lattices of genus $II_{b^+,b^-}(D)$ are isometric.*

Throughout this thesis, we will mainly be concerned with lattices of type $(n, 2)$, where n is a positive integer. So the first two conditions of the last theorems will always be satisfied. In the next section we will see how to compute the signature of a discriminant form without constructing a suitable lattice first. This will, together with these theorems, help to determine all combinations of n and discriminant forms D such that $II_{n,2}(D)$ is non-empty. In conjunction with further restrictions as, for instance, a prescribed level, this leaves only few discriminant forms to be considered. Their genera in turn will meet the requirements of Theorem 1.23 and hence contain only one lattice. We will therefore mostly just write the genus when, strictly speaking, referring to one of these lattices.

1.4 Jordan decomposition

As a finite abelian group, every discriminant form D can be decomposed into p-primary components, i.e. $D = \bigoplus_{p \text{ prime}} D_p$, where $D_p = \{\gamma \in D : p^r\gamma = 0 \text{ for some } r \in \mathbb{Z}\}$ for all primes p. If p_1, p_2 are distinct primes and $\gamma, \beta \in D$ are of order p_1^r, p_2^s, respectively, we can choose $a, b \in \mathbb{Z}$ such that $ap_1^r + bp_2^s = 1$ and

$$\gamma\beta = (ap_1^r + bp_2^s)\gamma\beta = (ap_1^r\gamma)\beta + \gamma(bp_2^s\beta) = 0 \bmod 1.$$

Hence $D = \bigoplus_{p \text{ prime}} D_p$ is indeed an orthogonal direct sum (in particular, the summands D_p are themselves discriminant forms). It turns out that each of the components D_p again can be written as an orthogonal direct sum, where every summand is, as a group, isomorphic to $(\mathbb{Z}/p^r\mathbb{Z})^{n_{p^r}}$ for some positive integers r, n_{p^r}. Such a summand is called a *(p-adic) Jordan component of exponent p^r and rank n_{p^r}* and the whole sum is a *Jordan decomposition* of D. Finally, each Jordan component of exponent q, where q is some prime power, decomposes into *indecomposable Jordan components* of some minimal rank and, of course, the same exponent. For more details on the proof of these and the following assertions on Jordan decompositions and for further references, see e.g. [CS98, Section 15.4].

In order to understand the definition of the different Jordan components below, first recall the *Legendre symbol*: for every odd prime number p and every integer a relatively prime to p, let $\left(\frac{a}{p}\right) = 1$ if a is a square mod p, i.e. $a = x^2 \bmod p$ for some $x \in \mathbb{Z}$, and $\left(\frac{a}{p}\right) = -1$ otherwise.

Let l, p be distinct odd prime numbers and $a, b \in \mathbb{Z}$ relatively prime to p. Since every polynomial of degree 2 over the finite field $\mathbb{F}_p$ has either 2 zeros in $\mathbb{F}_p$ or none, $(\mathbb{F}_p^\times)^2$ has index 2 in $\mathbb{F}_p^\times$, which is why we get $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ and $\left(\frac{a}{p}\right) = a^{(p-1)/2} \bmod p$. Besides, we have *Gauss's reciprocity law* [Neu99, Theorem 8.6]:

$$\left(\frac{l}{p}\right)\left(\frac{p}{l}\right) = (-1)^{\frac{p-1}{2}\frac{l-1}{2}}, \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}, \left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

This definition can be extended as follows.

Definition 1.24. Let $a, b \in \mathbb{Z}$. If $(a, b) > 1$, set the *Kronecker symbol* (also *extended Jacobi-Legendre symbol*) $\left(\frac{a}{b}\right)$ to 0. Otherwise, if b is an odd prime, $\left(\frac{a}{b}\right)$ is just defined as the usual Legendre symbol. Next, set

$$\left(\frac{a}{2}\right) = \begin{cases} 1 & \text{if } a = \pm 1 \bmod 8, \\ -1 & \text{if } a = \pm 3 \bmod 8, \end{cases}$$
$$\left(\frac{a}{-1}\right) = \begin{cases} 1 & \text{if } a \geq 0, \\ -1 & \text{if } a < 0, \end{cases}$$
$$\left(\frac{\pm 1}{0}\right) = 1.$$

Finally, if $b = (-1)^{\lambda_0} p_1^{\lambda_1} \cdots p_k^{\lambda_k}$ with primes $p_1, \dots, p_k$ and $\lambda_0, \dots, \lambda_k \geq 0$, let

$$\left(\frac{a}{b}\right) = \left(\frac{a}{-1}\right)^{\lambda_0} \left(\frac{a}{p_1}\right)^{\lambda_1} \cdots \left(\frac{a}{p_k}\right)^{\lambda_k}.$$

Note that this new symbol is multiplicative in both entries, i.e.

$$\left(\frac{ac}{bd}\right) = \left(\frac{a}{b}\right)\left(\frac{a}{d}\right)\left(\frac{c}{b}\right)\left(\frac{c}{d}\right) \quad \text{for all } a, b, c, d \in \mathbb{Z}.$$

Now we can list the different Jordan components in the usual manner of papers on the topic of automorphic products like [Sch09, Bor00].

- For a power $q > 1$ of an odd prime p, the nontrivial p-adic Jordan components of exponent q and rank $m \geq 1$ are denoted by $q^{\pm m}$. The indecomposable ones are those of rank 1. Such an indecomposable Jordan component $q^{\pm 1}$ is generated by an element γ with $q \cdot \gamma = 0$ and $\gamma^2/2 = a/q \bmod 1$ where a is some integer for which $\left(\frac{2a}{p}\right) = \pm 1$. These components have level q. The sign $\pm$ before the rank is also called the *sign* of the component.

- For a power $q > 1$ of 2, the nontrivial even 2-adic Jordan components of exponent q always have even rank. They are given by $q_{II}^{\pm 2m}$ where $m \geq 1$. The indecomposable ones are those of rank 2. Both q_{II}^{+2} an q_{II}^{-2} are generated by two elements γ, δ that satisfy $q \cdot \gamma = q \cdot \delta = 0$, $\gamma\delta = 1/q \bmod 1$ and $\gamma^2/2 = \delta^2/2 = 0 \bmod 1$ or $\gamma^2/2 = \delta^2/2 = 1/q \bmod 1$, respectively. They have level q, too.

- For a power $q > 1$ of 2, the nontrivial odd 2-adic Jordan components of exponent q and rank $m \geq 1$ are $q_t^{\pm m}$ with some $t \in \mathbb{Z}$. The indecomposable ones are $q_t^{\pm 1}$ where the *subscript* t satisfies $\left(\frac{t}{2}\right) = \pm 1$. By the rules for summation of Jordan components given below, this implies $t = 0$ or $t = \pm 2 \bmod 8$ for q_t^{+2} and $t = 4$ or $t = \pm 2 \bmod 8$ for q_t^{-2}, whereas for rank $m > 2$ the only restriction is $t = m \bmod 2$. The indecomposable components $q_t^{\pm 1}$ are generated by an element γ such that $q \cdot \gamma = 0$ and $\gamma^2/2 = t/2q \bmod 1$ and thus have level $2q$. Note that different signs and subscripts mod 8 do not necessarily mean that two odd 2-adic Jordan components are not isomorphic (e.g. $2_1^{+1} = 2_5^{-1}$).

- To formulate certain definitions and theorems neatly, it is sometimes convenient to define also Jordan components containing only one element. Let p be an odd prime and $q^{\pm m}$ such a trivial p-adic Jordan component. Then $q = 1$ or $m = 0$. For $m = 0$ we exclude the sign $-$, for $q = 1$ no restriction on the sign is assumed. Trivial 2-adic Jordan components are by convention always even and, again, $q_{II}^{\pm 0} = q_{II}^{+0}$.

For a sum $q^{\pm m} \oplus (q')^{\pm m'}$ one commonly writes $q^{\pm m}(q')^{\pm m'}$. If Jordan components with a common exponent are summed up, the result is a component of the same exponent, where the rank is the sum of the individual ranks and the sign the product of the individual signs. When furthermore working with odd 2-adic Jordan components, subscripts add up.

According to the structure theorem for finitely generated abelian groups, in the decomposition of such groups at least the orders (and thereby the isomorphy classes) of the involved cyclic subgroups are unique. As opposed to this, hardly anything about the Jordan decomposition of a discriminant form is unique in any sense. Indeed, if any two Jordan decompositions can be transformed into each other using the summation rules just given, they describe the same discriminant forms. For example, we have $2_{II}^{-2} 2_{II}^{-2} = 2_{II}^{+4} = 2_{II}^{+2} 2_{II}^{+2}$. Nonetheless one can show that the following notions, whose definitions rely on the Jordan decomposition, are well-defined.

Definition 1.25. Let p be an odd prime, J a Jordan component. The *p-excess* of J is defined by

$$p\text{-excess}(J) = \begin{cases} m(q-1) \bmod 8 & \text{if } J = q^{\pm m} \text{ is } p\text{-adic and } q \text{ a square or } \pm = +, \\ m(q-1) + 4 \bmod 8 & \text{if } J = q^{-m} \text{ is } p\text{-adic and } q \text{ not a square}, \\ 0 \bmod 8 & \text{if } J \text{ is not } p\text{-adic}. \end{cases}$$

For a general discriminant form D with Jordan decomposition $J_1 \oplus \cdots \oplus J_k$ we define $p\text{-excess}(D) = \sum_{i=1}^{k} p\text{-excess}(J_i)$ and $\gamma_p(D) = e(-p\text{-excess}(D)/8)$, introducing at the same time the abbreviation $e = \exp(2\pi i \,\cdot)$.

Definition 1.26. Let J be a Jordan component. The *oddity* of J is defined by

$$\text{oddity}(J) = \begin{cases} 0 \bmod 8 & \text{if } J = q_{II}^{\pm m} \text{ is even 2-adic and } q \text{ a square or } \pm = +, \\ 4 \bmod 8 & \text{if } J = q_{II}^{-m} \text{ is even 2-adic and } q \text{ not a square}, \\ t \bmod 8 & \text{if } J = q_t^{\pm m} \text{ is odd 2-adic and } q \text{ a square or } \pm = +, \\ t + 4 \bmod 8 & \text{if } J = q_t^{-m} \text{ is odd 2-adic and } q \text{ not a square}, \\ 0 \bmod 8 & \text{if } J \text{ is not 2-adic}. \end{cases}$$

For a general discriminant form D with Jordan decomposition $J_1 \oplus \cdots \oplus J_k$ we define $\text{oddity}(D) = \sum_{i=1}^{k} \text{oddity}(J_i)$ and $\gamma_2(D) = e(\text{oddity}(D)/8)$.

So if a Jordan decomposition of some discriminant form is known, these quantities can be computed and inserted in the following theorem. This is the method mentioned in the previous section to determine the signature without having to construct a lattice.

Theorem 1.27 (Oddity formula)**.** *Let D be a discriminant form. Then*

$$\operatorname{sign}(D) + \sum_{p\geq 3 \text{ prime}} p\text{-excess}(D) = \operatorname{oddity}(D) \bmod 8.$$

Obviously this is equivalent to the *product formula*

$$\prod_{p \text{ prime}} \gamma_p(D) = e(\operatorname{sign}(D)/8),$$

which is proved as Lemma 3.3.4 in [Cas78]. There are quite a few differences to the notation and conventions chosen here, but at least these reveal the origin of Definitions 1.25 and 1.26. (If, as in [CS98, Section 15.5], -1 is regarded as a prime and $\operatorname{sign}(D)$ as (-1)-excess(D), this can be formulated even more shortly as $\prod_{p \text{ prime}} \gamma_p(D) = 1$.) Let us apply this to the examples from Section 1.2.

Example 1.28. The discriminant form of the hyperbolic plane $II_{1,1}$ (which can be identified with its genus by Theorem 1.23) is trivial, hence there is no Jordan decomposition to be done. We see that, also in general, Theorem 1.27 implies one direction of Theorem 1.18.

As shown in Example 1.15, $A_2'/A_2 = \mathbb{Z}/3\mathbb{Z}$. Thus $A_2 \in II_{2,0}(3^{\pm 1})$ and it remains to compute the sign. As 3 is not a square, this can be done using the oddity formula, which yields $3\text{-excess}(A_2'/A_2) = -\operatorname{sign}(A_2'/A_2) = 6 \bmod 8$ and therefore $A_2 \in II_{2,0}(3^{-1})$. Alternatively, as a generator of A_2'/A_2 take the residue γ of $(\frac{2}{3}, \frac{1}{3})^\mathsf{T}$. Then $\gamma^2/2 = \frac{1}{3}$ and the sign is confirmed as $\left(\frac{2\cdot 1}{3}\right) = -1$.

Since there are no odd 2-adic Jordan components of level 2, by Example 1.15 we get $D_4'/D_4 = 2_{II}^{\pm 2}$. Again, Theorem 1.27 is the fast way to compute that $\operatorname{oddity}(D_4'/D_4) = \operatorname{sign}(D_4'/D_4) = 4 \bmod 8$ and consequently $D_4 \in II_{4,0}(2_{II}^{-2})$. The same result could be obtained by explicit construction of a basis of D_4' whose elements are pairwise distinct mod D_4. Here the inverse Gram matrix computed in Example 1.12 comes in useful. Afterwards one could deduce the sign from the fact that three basis elements have norm $\frac{1}{2}$.

Before concluding this chapter, some final definitions concerning discriminant forms are necessary. They will prove important when it comes to explicitly computing the objects that will be introduced in Chapter 3. Let D be a discriminant form, c an integer. As for every abelian group, multiplication by c is a homomorphism from D to itself. Denote by D_c its kernel and by D^c its image. Then we have an exact sequence

$$0 \to D_c \to D \to D^c \to 0$$

and furthermore

$$\begin{aligned}(D^c)^\perp &= \{\gamma \in D : \gamma\beta = 0 \bmod 1 \text{ for all } \beta \in D^c\}\\ &= \{\gamma \in D : c\gamma\beta = 0 \bmod 1 \text{ for all } \beta \in D\}\\ &= D_c\end{aligned}$$

since the bilinear form is non-degenerate. So D^c is the orthogonal complement of D_c. Next, define

$$D^{c*} = \{\alpha \in D : c\gamma^2/2 + \alpha\gamma = 0 \bmod 1 \text{ for all } \gamma \in D_c\}.$$

Proposition 1.29. [Sch09, Proposition 2.1] *D^{c*} is a coset of D^c.*

Proof. Let $\varphi : D_c \to \mathbb{Q}/\mathbb{Z} : \gamma \mapsto c\gamma^2/2$. Then

$$\varphi(\gamma + \beta) = c\gamma^2/2 + c\beta^2/2 + (c\gamma)\beta = \varphi(\gamma) + \varphi(\beta)$$

for all $\gamma, \beta \in D_c$. Hence $\varphi \in \mathrm{Hom}(D_c, \mathbb{Q}/\mathbb{Z})$ and $\varphi = (-\alpha, \cdot)$ for some $\alpha \in D$ (cf. the explanations after Definition 1.14). So $\alpha \in D^{c*}$ and any $\beta \in D$ belongs to D^{c*} if and only if $\beta - \alpha \in (D_c)^\perp = D^c$. □

In particular, we even have $D^{c*} = D^c$ if D has a Jordan decomposition that does not contain any odd 2-adic component [Sch09, p. 6].

Proposition 1.30. [Sch09, Proposition 2.2] *Let x_c be a representative of D^{c*}. For $\alpha \in D^{c*}$ choose $\gamma \in D$ such that $\alpha = x_c + c\gamma$. Then $c\gamma^2/2 + x_c\gamma \bmod 1$ is independent of γ.*

Proof. Let $\mu \in D$ such that $\alpha = x_c + c\mu$, too. Then $\gamma - \mu \in D_c$ and

$$\begin{aligned} c\gamma^2/2 + x_c\gamma - c\mu^2/2 - x_c\mu &= c\left(((\gamma-\mu)+\mu)^2/2 - \mu^2/2\right) + x_c(\gamma-\mu) \\ &= (c(\gamma-\mu))\,\mu + c(\gamma-\mu)^2/2 + x_c(\gamma-\mu) = 0 \bmod 1 \end{aligned}$$

by definition of D^{c*}. □

For a fixed representative x_c of D^{c*} this allows us to introduce the notation $\alpha_c^2/2 \coloneqq c\gamma^2/2 + x_c\gamma \bmod 1$, where γ is any element of D satisfying $\alpha = x_c + c\gamma$.

Chapter 2

Scalar valued modular forms

This chapter shall turn away from the topic of lattices and give a short overview of the theory of classical (scalar valued) modular forms. They are also known as elliptic modular forms and form a vast branch of active research. They have been attracting enormous attention since it was discovered that many a number-theoretic theorem or conjecture can be expressed in terms of Fourier coefficients of modular forms. There is also an important link to elliptic curves, subject of great progress in the last decades, and from there to Fermat's Last Theorem. In our case, the theory will be combined later on with the subject matter of Chapter 1 to a multidimensional one. Nonetheless, the most promising approach will then still be to try and break things down to scalar valued modular forms. Recommendable textbooks on the topic include [DS05, Miy06, Kna92], which also served as basis for this chapter.

2.1 Congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$

Denote by $\bar{\mathbb{C}}$ the *Riemann sphere* $\mathbb{C}\cup\{\infty\}$, where the element ∞ is usually thought of as lying infinitely far away and treated in computations as $\frac{a}{0}$ for any non-zero $a \in \mathbb{C}$, and let

$$\mathrm{GL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, ad - bc \neq 0 \right\}.$$

Endowed with the topology generated by the set

$$\left\{U \subseteq \bar{\mathbb{C}} : U \text{ is open in } \mathbb{C}\right\} \cup \left\{U \subseteq \bar{\mathbb{C}} : \bar{\mathbb{C}} \setminus U \text{ is compact in } \mathbb{C}\right\},$$

$\bar{\mathbb{C}}$ becomes a compact Riemann surface [Fre14]. For every $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$, the *fractional linear transformation*

$$\bar{\mathbb{C}} \to \bar{\mathbb{C}} : \tau \mapsto M\tau := \frac{a\tau + b}{c\tau + d}$$

is a biholomorphic map. Two elements of $\mathrm{GL}_2(\mathbb{C})$ yield the same transformation if and only if they differ by a scalar factor. Elementary calculations show that $(MN)\tau = M(N\tau)$ for all $M, N \in \mathrm{GL}_2(\mathbb{C})$, $\tau \in \bar{\mathbb{C}}$, i.e. we have a group action of

$\mathrm{GL}_2(\mathbb{C})$ on $\bar{\mathbb{C}}$. Now consider the *upper half plane* $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ and the subgroup

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

of $\mathrm{GL}_2(\mathbb{C})$, which is, in this context, usually referred to as the *modular group*. It is the free group generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

modulo the relations $S^2 = -1, (ST)^3 = -1$ or, strictly speaking, the relations S^4 and $S^{-2}(ST)^3$ [Alp93]. Then $\mathrm{Im}(M\tau) = |j(M,\tau)|^{-2}\,\mathrm{Im}(\tau) > 0$ with the *factor of automorphy* $j\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau\right) \coloneqq c\tau + d$ for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathbb{H}$. So the above restricts to a group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. Similarly, there is also a restriction on the set $\mathbb{Q} \cup \{\infty\}$, which can easily be shown to be (even triply) transitive. A certain class of subgroups of $\mathrm{SL}_2(\mathbb{Z})$ is of special interest in the theory of modular forms.

Definition 2.1. For every positive integer N the *principal congruence subgroup of level* N is the group

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d = 1 \bmod N, b, c = 0 \bmod N \right\}.$$

Any subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* if $\Gamma(N) \subseteq \Gamma$ for some N, which is then called the *level* of Γ. Further, for every level N some specific congruence subgroups are given by

$$\Gamma_0(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c = 0 \bmod N \right\}$$

and

$$\Gamma_1(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a, d = 1, c = 0 \bmod N \right\}.$$

Every principal congruence subgroup $\Gamma(N)$ is the kernel of the canonical homomorphism from $\mathrm{SL}_2(\mathbb{Z})$ onto the finite group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. This shows that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] < \infty$ and consequently every other congruence subgroup has finite index in $\mathrm{SL}_2(\mathbb{Z})$, too. More precisely, counting the elements of $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and applying analogous methods to $\Gamma_0(N), \Gamma_1(N)$ yields the following.

Theorem 2.2. [DS05, pp. 13f.] *Let N be a positive integer. Then*

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{\substack{p \text{ prime} \\ p \mid N}} \left(1 - \frac{1}{p^2}\right),$$

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)] = N^2 \prod_{\substack{p \text{ prime} \\ p \mid N}} \left(1 - \frac{1}{p^2}\right)$$

and

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{\substack{p \text{ prime} \\ p \mid N}} \left(1 + \frac{1}{p}\right).$$

Since the full modular group acts on $\mathbb{Q} \cup \{\infty\}$, so does every congruence subgroup, albeit of course not necessarily transitively. The number of orbits is obviously bounded from above by the index in $\mathrm{SL}_2(\mathbb{Z})$. For the special cases emphasized above, it is computed exactly in [DS05, Section 3.8].

Definition 2.3. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The orbits of points in $\mathbb{Q} \cup \{\infty\}$ under Γ are called *cusps* of Γ. Their number will be denoted by $\varepsilon_\infty(\Gamma)$.

Theorem 2.4. *Let ϕ be Euler's totient function and N a positive integer. Then we have*

$$\begin{aligned} \varepsilon_\infty(\Gamma(N)) &= \frac{1}{2} N^2 \prod_{\substack{p \text{ prime} \\ p \mid N}} \left(1 - \frac{1}{p^2}\right) \ \textit{if } N > 2, \\ \varepsilon_\infty(\Gamma_1(N)) &= \frac{1}{2} \sum_{\delta \mid N} \phi(\delta)\phi(\tfrac{N}{\delta}) \ \textit{if } N = 3 \textit{ or } N > 4, \\ \varepsilon_\infty(\Gamma_0(N)) &= \sum_{\delta \mid N} \phi\big((\delta, \tfrac{N}{\delta})\big) \ \textit{if } N > 2 \end{aligned}$$

(when summing over divisors of some number, these are always meant to be positive unless noted otherwise) and further

$$\begin{aligned} \varepsilon_\infty(\Gamma(2)) &= 3, \\ \varepsilon_\infty(\Gamma_0(2)) = \varepsilon_\infty(\Gamma_1(2)) &= 2, \\ \varepsilon_\infty(\Gamma_1(4)) &= 3. \end{aligned}$$

2.2 Modular functions

Modular forms are certain complex-valued functions on the upper half plane $\mathbb{H}$. They interact with fractional linear transformations in a specific manner, depending on their so-called weight and the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ in question, and satisfy some enhanced holomorphy conditions. Depending on whether the latter are somewhat weakened or strengthened, there are also the notions of weakly modular functions, almost holomorphic modular forms and cusp forms. Let us first make some more definitions and conventions to keep the notation short and the theory as general as eventually necessary.

Definition 2.5. Let Γ be a congruence subgroup of level N. A group homomorphism $\chi : \Gamma \to \mathbb{C}^\times$ is a *character* mod N *of* Γ if $\chi(M) = 1$ for all $M \in \Gamma(N)$.

Proposition 2.6. *Let Γ be a congruence subgroup of level N, χ a character* mod *N of Γ. Then χ has finite order. In particular, χ is* unitary, *i.e.* $|\chi(M)| = 1$ *for all $M \in \Gamma$.*

Proof. We have $[\ker(\chi) : \Gamma] \leq [\Gamma(N) : \mathrm{SL}_2(\mathbb{Z})] < \infty$. So the image of χ is a finite subgroup of $\mathbb{C}^\times$ and its order a multiple of $\mathrm{ord}(\chi)$. □

Definition 2.7. For $k \in \mathbb{Z}$, $M \in \mathrm{SL}_2(\mathbb{Z})$ the *weight-k-operator* on the space of functions from $\mathbb{H}$ to $\mathbb{C}$ is defined by

$$f[M]_k(\tau) = j(M,\tau)^{-k} f(M\tau)$$

for all functions $f : \mathbb{H} \to \mathbb{C}$ and all $\tau \in \mathbb{H}$. If the weight k is understood, we will write $f|_M$ instead of $f[M]_k$.

It is easily verified that for every $K, M \in \mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathbb{H}$ the factor of automorphy behaves like $j(KM,\tau) = j(K, M\tau)j(M,\tau)$. Therefore also $f|_{KM} = (f|_K)|_M$ for every function $f : \mathbb{H} \to \mathbb{C}$ and fixed $k \in \mathbb{Z}$.

Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and h-periodic for some $h > 0$, that is, $f(\tau+h) = f(\tau)$ for all $\tau \in \mathbb{H}$. Then f has a Fourier expansion: $f(\tau) = \sum_{m=-\infty}^{\infty} a_m q_h^m$ for all $\tau \in \mathbb{H}$, where the sequence $(a_m)_{m\in\mathbb{Z}}$ of Fourier coefficients is uniquely determined and the convergence locally uniform. Here the common abbreviation $q_h := e(\tau/h) = \exp(2\pi i\tau/h)$ has been used, whence also the name *q_h-expansion*. If it exists, the real number $\min\{m/h : a_m \neq 0\}$ is called the *order* of f at ∞, denoted by $\mathrm{ord}_\infty(f)$. Otherwise $\mathrm{ord}_\infty(f) = \infty$ or $\mathrm{ord}_\infty(f) = -\infty$, depending on whether $a_m \neq 0$ for no $m \in \mathbb{Z}$ or for infinitely many $m < 0$. The function f is called *meromorphic at* ∞ if $\mathrm{ord}_\infty(f) > -\infty$. It is *holomorphic at* ∞ if $\mathrm{ord}_\infty(f) \geq 0$ and *zero at* ∞ if also $\mathrm{ord}_\infty(f) > 0$.

Definition 2.8. Let k be an integer, Γ a congruence subgroup of level N, χ a character mod N of Γ and $f : \mathbb{H} \to \mathbb{C}$ meromorphic. Then f is *weakly modular of weight k and character χ with respect to* Γ if $f[M]_k = \chi(M)f$ for all $M \in \Gamma$.

Under these assumptions, for every $M \in \mathrm{SL}_2(\mathbb{Z})$ the function $f[M]_k$ is weakly modular with respect to $\Gamma(N) \subseteq M^{-1}\Gamma M$. As it happens, $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(N)$ and the corresponding fractional linear transformation is the translation $\mathbb{H} \to \mathbb{H} : \tau \mapsto \tau + N$. So if f is even holomorphic, there is a Fourier expansion in powers of q_N. In fact, in many cases it suffices to expand in powers of q_δ for some positive divisor δ of N. For instance, if χ is trivial and $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$, at ∞ there is an expansion in powers of q since $T \in \Gamma$.

For every cusp s of Γ the number $\mathrm{ord}_s(f) := \mathrm{ord}_\infty(f[M]_k)$, where M is an element of $\mathrm{SL}_2(\mathbb{Z})$ such that $M\infty = s$, is called the *order of f at s*. It is independent of the representative for s and of the choice of M since M is unique up to some matrix of the form $\pm T^m = \left(\begin{smallmatrix} \pm 1 & \pm m \\ 0 & \pm 1 \end{smallmatrix}\right)$, $m \in \mathbb{Z}$. The *value* of f at s, however, need not be well-defined. Note also that, unlike the classical order $\mathrm{ord}_\tau(f)$ of any meromorphic function $f : \mathbb{H} \to \mathbb{C}$ at a point τ inside $\mathbb{H}$, the definition of the order at a cusp already involves the weight and therefore requires weak modularity.

Definition 2.9. In Definition 2.8, if additionally f is holomorphic on $\mathbb{H}$ and $f[K]_k$ is meromorphic at ∞ for all $K \in \mathrm{SL}_2(\mathbb{Z})$, f is an *almost holomorphic modular form of weight k and character χ for* Γ. If the functions $f[K]_k$ are even holomorphic or zero at ∞, we call f a *modular form* or *cusp form of weight k and character χ for* Γ, respectively.

The complex vector spaces of almost holomorphic modular forms, (holomorphic) modular forms and cusp forms of weight k and character χ for Γ shall be denoted by $\mathcal{M}_k^!(\Gamma,\chi)$, $\mathcal{M}_k(\Gamma,\chi)$, and $\mathcal{S}_k(\Gamma,\chi)$, respectively, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ or trivial χ can be omitted. A principal reason why the theory of modular forms is so powerful is that some of these spaces are actually finite-dimensional, which often makes it easy to establish non-obvious identities of modular forms. It turns out that the holomorphy conditions imposed on $\mathcal{M}_k(\Gamma,\chi)$ are crucial to that end. More precisely, the set $\Gamma\backslash\mathbb{H}$ of orbits under the action of Γ on $\mathbb{H}$ becomes a compact Riemann surface $X(\Gamma)$ when the cusps are added [DS05, Section 2.4]. Then the Riemann-Roch Theorem can be used to obtain precise dimension formulae from the topological data of $X(\Gamma)$, provided the function spaces in question prescribe holomorphy outside a fixed finite set of points and lower bounds on the order inside. Some of these formulae will be stated below, but first let us consider some basic examples of modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

Obviously every constant function on $\mathbb{H}$ is a modular form of weight 0. We will see later that indeed $\mathcal{M}_0 = \mathbb{C}$. For every integer $k > 2$ and $\tau \in \mathbb{H}$, define

$$G_k(\tau) = \sum_{\substack{(c,d)\in\mathbb{Z}^2\\(c,d)\neq(0,0)}} \frac{1}{(c\tau+d)^k}.$$

Since $k > 2$, the series converges absolutely locally uniformly on $\mathbb{H}$ and thus defines a holomorphic function. Easy calculations show that $\lim_{\mathrm{Im}(\tau)\to\infty} G_k(\tau) = \sum_{d\in\mathbb{Z}\backslash\{0\}} d^{-k} < \infty$. Furthermore G_k is weakly modular of weight k and trivial character with respect to the full modular group. So $G_k \in \mathcal{M}_k$ for all $k > 2$. For odd k the summands obviously cancel in accordance with the fact that $\mathcal{M}_k(\Gamma) = \{0\}$ for odd k whenever $-1 \in \Gamma$: then

$$-f(\tau) = j(-1,\tau)^k f((-1)\tau) = f|_{-1}(\tau) = f(\tau)$$

for all $f \in \mathcal{M}_k(\Gamma)$, $\tau \in \mathbb{H}$. When k is even, G_k can be normalized to the *Eisenstein series of weight k* (*for the full modular group*; there is also a more general theory of Eisenstein series for other subgroups). It has rational Fourier coefficients, the constant one being 1:

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)} = 1 - \frac{2k}{B_k}\sum_{m=1}^{\infty}\sigma_{k-1}(m)q^m,$$

where ζ is the Riemann zeta function, $\sigma_{k-1}(m) = \sum_{\delta|m}\delta^{k-1}$ and the *Bernoulli numbers B_k* are defined by the formal power series identity

$$\frac{x}{\exp(x)-1} = \sum_{k=0}^{\infty} B_k\frac{x^k}{k!}.$$

Once this q-expansion and $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ have been computed, one directly concludes that $E_4^3 - E_6^2$ is a nontrivial cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. Its multiple $\Delta = \frac{64\pi^{12}}{27}(E_4^3 - E_6^2)$ is called the *discriminant function*.

Theorem 2.10. [DS05, Thm. 3.5.2] *Let $k \in \mathbb{Z}$. Then*

$$\dim(\mathcal{M}_k) = \begin{cases} 0 & \text{if } k < 0 \text{ or } k \text{ is odd,}\\ \left\lfloor \frac{k}{12}\right\rfloor & \text{if } k > 0 \text{ and } k = 2 \bmod 12,\\ \left\lfloor \frac{k}{12}\right\rfloor + 1 & \text{otherwise}\end{cases}$$

and

$$\dim(\mathcal{S}_k) = \begin{cases} 0 & \textit{if } k < 12 \textit{ or } k \textit{ is odd,} \\ \left\lfloor \frac{k}{12} \right\rfloor - 1 & \textit{if } k > 12 \textit{ and } k = 2 \bmod 12, \\ \left\lfloor \frac{k}{12} \right\rfloor & \textit{otherwise.} \end{cases}$$

Corollary 2.11. *For every $k \in \mathbb{Z}$, a basis of $\mathcal{M}_k$ is given by*

$$\left\{ E_4^a E_6^b : a, b \in \mathbb{Z}, a, b \geq 0 \textit{ and } 4a + 6b = k \right\}.$$

In particular, the graded ring of modular forms for $\mathrm{SL}_2(\mathbb{Z})$,

$$\mathcal{M}(\mathrm{SL}_2(\mathbb{Z})) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})),$$

is equal to the polynomial ring $\mathbb{C}[E_4, E_6]$.

Proof. If the dimensions of these spaces are assumed to be already known, the proof of [Ste07, Theorem 2.17] can be shortened considerably. We proceed by induction on the weight. For $k < 0$ and odd k the statement is trivial. For the other cases, first observe that

$$\dim(\mathcal{M}_k) = \left| \left\{ (a, b) \in \mathbb{Z}^2 : a, b \geq 0 \text{ and } 4a + 6b = k \right\} \right|.$$

For $k < 12$ this is easily checked. If $k \geq 12$, the set on the right hand side contains exactly one pair (a, b) such that $b \in \{0, 1\}$. The others can be mapped bijectively to the set corresponding to $k - 12$, namely via $(a, b) \mapsto (a, b - 2)$. As by Theorem 2.10 also $\dim(\mathcal{M}_k) = \dim(\mathcal{M}_{k-12}) + 1$, this confirms the observation.

Thus it only remains to show that the set generates $\mathcal{M}_k$. For $k < 12$ this is again clear. The cusp form Δ does not vanish on all of $\mathbb{H}$, so $\mathcal{M}_{k-12} \to \mathcal{S}_k : f \mapsto \Delta \cdot f$ is injective and consequently an isomorphism for dimension reasons. Applying the induction hypothesis to $\mathcal{M}_{k-12}$ then finishes the proof since Δ is a polynomial in E_4 and E_6, too, $\dim(\mathcal{M}_k) - \dim(\mathcal{S}_k) = 1$ and $E_4^a E_6^b \in \mathcal{M}_k \setminus \mathcal{S}_k$ for suitable $a, b \in \mathbb{Z}$. □

There is also a dimension formula for the case of a general congruence subgroup [DS05, Theorem 3.5.1]. It does not simplify significantly for $\Gamma_0(N)$, $N \in \mathbb{N}$. But for $\Gamma(N)$ and $\Gamma_1(N)$, which fortunately are the only cases we will have to deal with, we get the following formulae.

Theorem 2.12. [DS05, Section 3.9] *Let $N > 1$ be an integer, $k \in \mathbb{Z} \setminus \{1\}$ and define*

$$d_N = \frac{1}{2} N^3 \prod_{\substack{p \text{ prime} \\ p \mid N}} \left(1 - \frac{1}{p^2} \right)$$

if $N > 2$. Then

$$\dim\big(\mathcal{M}_k(\Gamma(N))\big) = \dim\big(\mathcal{S}_k(\Gamma(N))\big) = \dim\big(\mathcal{M}_k(\Gamma_1(N))\big) = \dim\big(\mathcal{S}_k(\Gamma_1(N))\big) = 0$$

if $k < 0$. Otherwise

$$\dim\big(\mathcal{M}_k(\Gamma(N))\big) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{(k-1)d_N}{12} + \frac{d_N}{2N} & \text{if } k \geq 2, \end{cases}$$

$$\dim\big(\mathcal{S}_k(\Gamma(N))\big) = \begin{cases} 0 & \text{if } k = 0, \\ 1 + \frac{d_N(N-6)}{12N} & \text{if } k = 2, \\ \frac{(k-1)d_N}{12} - \frac{d_N}{2N} & \text{if } k \geq 3 \end{cases}$$

for $N > 2$ and

$$\dim\big(\mathcal{M}_k(\Gamma_1(N))\big) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{(k-1)d_N}{12} + \frac{1}{4}\sum_{\delta | N} \phi(\delta)\phi(\frac{N}{\delta}) & \text{if } k \geq 2, \end{cases}$$

$$\dim\big(\mathcal{S}_k(\Gamma_1(N))\big) = \begin{cases} 0 & \text{if } k = 0, \\ 1 + \frac{d_N}{12N} - \frac{1}{4}\sum_{\delta | N} \phi(\delta)\phi(\frac{N}{\delta}) & \text{if } k = 2, \\ \frac{(k-1)d_N}{12} - \frac{1}{4}\sum_{\delta | N} \phi(\delta)\phi(\frac{N}{\delta}) & \text{if } k \geq 3 \end{cases}$$

for $N > 4$. Further

$$\dim\big(\mathcal{M}_k(\Gamma(2))\big) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ \frac{k+2}{2} & \text{if } k \text{ is even}, \end{cases}$$

$$\dim\big(\mathcal{S}_k(\Gamma(2))\big) = \begin{cases} 0 & \text{if } k \leq 2 \text{ or } k \text{ is odd}, \\ \frac{k-4}{2} & \text{if } k \geq 4 \text{ is even}, \end{cases}$$

$$\dim\big(\mathcal{M}_k(\Gamma_1(2))\big) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ \lfloor \frac{k}{4} \rfloor + 1 & \text{if } k \text{ is even}, \end{cases}$$

$$\dim\big(\mathcal{S}_k(\Gamma_1(2))\big) = \begin{cases} 0 & \text{if } k \leq 2 \text{ or } k \text{ is odd}, \\ \lfloor \frac{k}{4} \rfloor - 1 & \text{if } k \geq 4 \text{ is even}, \end{cases}$$

$$\dim\big(\mathcal{M}_k(\Gamma_1(3))\big) = \lfloor \tfrac{k}{3} \rfloor + 1,$$

$$\dim\big(\mathcal{S}_k(\Gamma_1(3))\big) = \begin{cases} 0 & \text{if } k \leq 2, \\ \lfloor \frac{k}{3} \rfloor - 1 & \text{if } k \geq 3, \end{cases}$$

$$\dim\big(\mathcal{M}_k(\Gamma_1(4))\big) = \begin{cases} \frac{k+2}{2} & \text{if } k \text{ is even}, \\ \frac{k+1}{2} & \text{if } k \text{ is odd}, \end{cases}$$

$$\dim\big(\mathcal{S}_k(\Gamma_1(4))\big) = \begin{cases} 0 & \text{if } k \leq 2, \\ \frac{k-4}{2} & \text{if } k \geq 4 \text{ is even}, \\ \frac{k-3}{2} & \text{if } k \text{ is odd}. \end{cases}$$

What we will precisely need are spaces of cusp forms for $\Gamma_1(N)$ with some character χ mod N. Then $\mathcal{S}_k(\Gamma_1(N))$ is only the special case for $\chi = 1$, whereas all of these are subspaces of $\mathcal{S}_k(\Gamma(N))$. The following theorem describes more accurately how the spaces $\mathcal{S}_k(\Gamma_1(N), \chi)$ are related to those considered in Theorem 2.12. For the cases that are treated in this thesis, we will later on have to construct concrete bases. Although Theorem 2.13 is not strictly necessary to verify these constructions, it should be illuminating with regard to what is happening in the background.

Theorem 2.13. *Let $k, N \in \mathbb{Z}$, $N > 0$. Then*

$$\mathcal{M}_k(\Gamma(N)) = \bigoplus_{j=0}^{N-1} \mathcal{M}_k(\Gamma_1(N), \chi_j) \quad \textit{and} \quad \mathcal{S}_k(\Gamma(N)) = \bigoplus_{j=0}^{N-1} \mathcal{S}_k(\Gamma_1(N), \chi_j),$$

where for each $j \in \{0, \ldots, N-1\}$ the character χ_j mod N of $\Gamma_1(N)$ is defined as

$$\chi_j : \Gamma_1(N) \to \mathbb{C}^\times : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto e\left(\tfrac{jb}{N}\right).$$

Proof. Clearly

$$\varphi : \Gamma_1(N) \to \mathbb{Z}/N\mathbb{Z} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b + N\mathbb{Z}$$

is a surjective homomorphism with kernel $\Gamma(N)$. So every character mod N of $\Gamma_1(N)$ is of the form $\psi \circ \varphi$, where ψ is a character of $\mathbb{Z}/N\mathbb{Z}$ and therefore equal to $e(\frac{j\cdot}{N})$ for some $j \in \mathbb{Z}$. Thus the χ_j are exactly the characters mod N of $\Gamma_1(N)$.

The rest of the proof is inspired by [Miy06, Lemma 4.3.1]. Since $\Gamma(N)$ is normal in $\Gamma_1(N)$, for every $M \in \Gamma_1(N)$, $K \in \Gamma(N)$ there is a $K' \in \Gamma(N)$ such that $M^{-1}K = K'M^{-1}$ and

$$(f|_{M^{-1}})|_K = f|_{M^{-1}K} = f|_{K'M^{-1}} = (f|_{K'})|_{M^{-1}} = f|_{M^{-1}}$$

for all $f \in \mathcal{M}_k(\Gamma(N))$. This shows that

$$\Gamma_1(N)/\Gamma(N) \times \mathcal{M}_k(\Gamma(N)) \to \mathcal{M}_k(\Gamma(N)) : (M\Gamma(N), f) \mapsto f|_{M^{-1}}$$

is a well-defined group action. In other words, we have a representation of the finite abelian group $\Gamma_1(N)/\Gamma(N)$ on the vector space $\mathcal{M}_k(\Gamma(N))$, which therefore decomposes into one-dimensional irreducible representations. But a one-dimensional representation is just the same as multiplication by a character. Along with the obvious fact that $\mathcal{S}_k(\Gamma(N))$ is a subrepresentation of $\mathcal{M}_k(\Gamma(N))$, this finishes the proof. □

2.3 Eta products and theta series

In order to construct the bases mentioned in the previous section, we need some more building blocks than just E_k and Δ. Accordingly the rest of this chapter is intended to provide the relevant information on two important classes of examples.

Definition 2.14. The *Dedekind eta function* is defined by

$$\eta(\tau) = q_{24} \prod_{m=1}^{\infty} (1 - q^m)$$

for all $\tau \in \mathbb{H}$.

Since $\sum_{m=1}^{\infty} q^m$ converges absolutely and compactly uniformly on $\mathbb{H}$, by a standard result from the theory of infinite products η is a holomorphic function on $\mathbb{H}$. Now obviously $\operatorname{ord}_\infty(\eta) = \frac{1}{24} > 0$ and we have $\eta(\tau) \neq 0$ and $\eta(T\tau) = \eta(\tau+1) = e(\frac{1}{24})\eta(\tau)$ for all $\tau \in \mathbb{H}$. The function also behaves nicely in combination with the other generator of the modular group, namely like $\eta(S\tau) = \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$ for all $\tau \in \mathbb{H}$, where the notation implies that the principal branch of the square root is used. A general transformation formula reads as follows.

Theorem 2.15. [Rad73, Section 74] *Let $\tau \in \mathbb{H}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$ or $c = 0$, $d > 0$. Then*

$$\eta(M\tau) = \varepsilon(M)\sqrt{c\tau + d}\,\eta(\tau),$$

where

$$\varepsilon(M) = \begin{cases} \left(\frac{d}{c}\right) e\big((-3c + bd(1-c^2) + c(a+d))/24\big) & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right) e\big((3d - 3 + ac(1-d^2) + d(b-c))/24\big) & \text{if } d \text{ is odd.} \end{cases}$$

Note that $M\tau = (-M)\tau$, so here the conditions on c and d are no real restriction. Concerning the construction of new modular forms, that already looks quite promising. Indeed we have the following result.

Theorem 2.16. *Let $N > 0$ be an integer and $r_\delta \in \mathbb{Z}$ for every positive divisor δ of N. If $\frac{N}{24}\sum_{\delta|N} r_\delta \delta$, $\frac{N}{24}\sum_{\delta|N} r_\delta/\delta$ and $k := \frac{1}{2}\sum_{\delta|N} r_\delta$ are integers, the function*

$$f : \mathbb{H} \to \mathbb{C} : \tau \mapsto \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$$

is an almost holomorphic modular form of weight k for $\Gamma_1(N)$ with character

$$\chi : \Gamma_1(N) \to \mathbb{C}^\times : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto e\Big(\frac{b}{24}\sum_{\delta|N} r_\delta \delta\Big).$$

An analogous statement for $\Gamma_0(N)$ was proved in [Bor00]. The proof of this version is taken from [Sch13] and will be presented below. It requires some minor preliminaries.

Proposition 2.17. *Let $f : \mathbb{H} \to \mathbb{C}$, Γ some subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and χ a character of Γ. Then $\{M \in \Gamma : f|_M = \chi(M)f\}$ is a subgroup of Γ.*

Proof. This is just the stabilizer of f under the group action

$$\Gamma \times \mathbb{C}^{\mathbb{H}} \to \mathbb{C}^{\mathbb{H}} : (M, g) \mapsto \chi(M) g|_{M^{-1}}$$

of Γ on the set of complex valued functions on $\mathbb{H}$. □

Lemma 2.18. *Let $N > 0$ be an integer. Then $\Gamma_1(N)$ is generated by the set*

$$\Omega = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) : c \geq 0, d = 1 \bmod 4 \right\}.$$

Proof. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N) \setminus \Omega$. Then $c = 0$ only if $a = d = -1$ and $N = 2$. In this case

$$M = -\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$$

is a product of elements of Ω. If $c \neq 0$, let $m \in \mathbb{Z}$ such that $a + mc > 0$ and $b + md$ is odd. Then

$$\begin{aligned} M &= \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a + mc & b + md \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ m'N & 1 \end{pmatrix}^{-1} \begin{pmatrix} a + mc & b + md \\ c + m'N(a + mc) & d + m'N(b + md) \end{pmatrix} \end{aligned}$$

for all $m' \in \mathbb{Z}$. Now choose $m' > 0$ such that $m'(b + md) = (1 - d)/N \bmod 4$ and $c + m'N(a + mc) \geq 0$. □

Lemma 2.19. *Let $a, b \in \mathbb{Z}$, $a > 0$ and b odd. Then $\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right)$ if $b = 1 \bmod 4$ and $\left(\frac{a}{b}\right) = \left(\frac{-b}{a}\right)$ otherwise.*

Proof. If a and b are not relatively prime, the statement is trivial. Otherwise write $a = p_1 \cdots p_\ell$ with prime numbers $p_1, \dots, p_\ell$ and $b = q_1 \cdots q_m$, where q_j is an odd prime or $q_j = -1$ for each $j \in \{1, \dots, m\}$. We may assume that there is a $j' \in \{1, \dots, m\}$ such that $q_1, \dots, q_{j'} = 3 \bmod 4$ and $q_{j'+1}, \dots, q_m = 1 \bmod 4$. Let $i \in \{1, \dots, \ell\}$, $j \in \{1, \dots, m\}$. Then by Gauss's reciprocity law

$$\left(\frac{p_i}{q_j}\right) = \begin{cases} \left(\frac{q_j}{p_i}\right)(-1)^{\frac{(p_i-1)(q_j-1)}{4}} = \left(\frac{q_j}{p_i}\right) & \text{if } j > j',\ p_i \text{ odd}, \\ \left(\frac{q_j}{p_i}\right)(-1)^{\frac{(p_i-1)(q_j-1)}{4}} = \left(\frac{q_j}{p_i}\right)(-1)^{\frac{p_i-1}{2}} = \left(\frac{-q_j}{p_i}\right) & \text{if } j \leq j',\ q_j \neq -1,\ p_i \text{ odd}, \\ (-1)^{\frac{q_j^2-1}{8}} = \left(\frac{q_j}{p_i}\right) & \text{if } p_i = 2, \\ 1 = \left(\frac{-q_j}{p_i}\right) & \text{if } q_j = -1. \end{cases}$$

Now suppose $b = 1 \bmod 4$. Then j' is even and

$$\begin{aligned} \left(\frac{a}{b}\right) &= \prod_{i=1}^{\ell} \prod_{j=1}^{m} \left(\frac{p_i}{q_j}\right) = \left(\prod_{\substack{1 \leq i \leq \ell \\ p_i = 2}} \prod_{j=1}^{m} \left(\frac{q_j}{p_i}\right) \right) \prod_{\substack{1 \leq i \leq \ell \\ p_i \neq 2}} \left(\prod_{j=1}^{j'} \left(\frac{-q_j}{p_i}\right) \right) \left(\prod_{j=j'+1}^{m} \left(\frac{q_j}{p_i}\right) \right) \\ &= \left(\frac{(-1)^{j'} b}{a}\right) = \left(\frac{b}{a}\right). \end{aligned}$$

This also yields

$$\left(\frac{a}{b}\right) = \left(\frac{a}{-1}\right) \left(\frac{a}{b}\right) = \left(\frac{a}{-b}\right) = \left(\frac{-b}{a}\right)$$

if $b = 3 \bmod 4$. □

Proposition 2.20. [Sch09, Proposition 6.2] *Let m be a positive integer, $\tau \in \mathbb{H}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$ or $c = 0$, $d > 0$. Suppose r, s, t are integers such that*

$$r, t > 0, \quad rt = m, \quad r \mid c \quad \text{and} \quad m \mid dr - cs.$$

Then

$$\sqrt{c\tau+d}^{-1}\eta\big(m(M\tau)\big)=\varepsilon(M')\frac{1}{\sqrt{t}}\eta\left(\frac{r\tau+s}{t}\right)$$

where ε *is defined as in Theorem 2.15 and*

$$M'=\begin{pmatrix} at & br-as \\ c/r & (dr-cs)/m \end{pmatrix}.$$

Moreover, we can always find such r,s,t *and* r,t *are uniquely determined by* $r=(m,c)$.

Proof. Define $A=\frac{1}{\sqrt{m}}\left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B=\frac{1}{\sqrt{m}}\left(\begin{smallmatrix} r & s \\ 0 & t \end{smallmatrix}\right)$. Then $AMB^{-1}=M'\in \mathrm{SL}_2(\mathbb{Z})$ and therefore, according to Theorem 2.15,

$$\begin{aligned}\eta\big(m(M\tau)\big)&=\eta\big(AM\tau\big)\\&=\eta\big(M'B\tau\big)\\&=\varepsilon(M')\sqrt{\frac{c}{r}\left(\frac{r\tau+s}{t}\right)+\frac{dr-cs}{m}}\,\eta\big(B\tau\big)\\&=\varepsilon(M')\frac{1}{\sqrt{t}}\sqrt{c\tau+d}\,\eta\left(\frac{r\tau+s}{t}\right).\end{aligned}$$

Further, we obviously must have $r\mid(m,c)$. Conversely, $(m,c)\mid(dr-cs)+cs=dr$, hence also $(m,c)\mid r$ since $(c,d)=1$. So in total, $r=(m,c)$ and $t=m/(m,c)$. Finally it remains to show that, when defining r and t like this, there is some $s\in\mathbb{Z}$ satisfying $m\mid dr-cs$. This relation is equivalent to $t\mid d-(c/r)s$. As $(t,c/r)=(m/(m,c),c/(m,c))=1$, we can just choose $s=d(c/r)^{-1}$, where the inverse is taken mod t. □

Proof of Theorem 2.16. It is clear that f is holomorphic on $\mathbb{H}$ and, by Proposition 2.20, that it is meromorphic at the cusps (provided this notion makes sense, i.e. f is weakly modular). Thus it remains to show that $(f|_M)(\tau)=\chi(M)f(\tau)$ for all $\tau\in\mathbb{H}$ and any $M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_1(N)$.

By Proposition 2.17 and Lemma 2.18 we may assume $c\geq 0$ and $d=1 \bmod 4$ (in particular $d=1>0$ if $c=0$). For every positive divisor δ of N, inserting $m=r=\delta$, $t=1$ and $s=0$ in Proposition 2.20 yields $M'=M'_\delta=\left(\begin{smallmatrix} a & b\delta \\ c/\delta & d \end{smallmatrix}\right)$ and further

$$\begin{aligned}f|_M(\tau)&=\prod_{\delta|N}(c\tau+d)^{-r_\delta/2}\eta\big(\delta(M\tau)\big)^{r_\delta}\\&=\prod_{\delta|N}\varepsilon(M'_\delta)^{r_\delta}\eta(\delta\tau)^{r_\delta}=\left(\prod_{\delta|N}\varepsilon(M'_\delta)^{r_\delta}\right)f(\tau).\end{aligned}$$

All M'_δ have odd lower right entry d. Hence

$$\begin{aligned}\prod_{\delta|N}\varepsilon(M'_\delta)^{r_\delta}&=\prod_{\delta|N}\left(\frac{c/\delta}{d}\right)^{r_\delta}e\left((3d-3+ac(1-d^2)/\delta+d(b\delta-c/\delta))/24\right)^{r_\delta}\\&=\left(\frac{c}{d}\right)^{2k}\left(\prod_{\delta|N}\left(\frac{\delta}{d}\right)^{r_\delta}\right)e\left(\frac{bd}{24}\sum_{\delta|N}r_\delta\delta+(a-d-ad^2)\frac{c}{24}\sum_{\delta|N}r_\delta/\delta+\frac{3(d-1)}{24}\sum_{\delta|N}r_\delta\right)\\&=\chi(M)\end{aligned}$$

as by assumption

$$\frac{b(d-1)}{24}\sum_{\delta|N} r_\delta\delta,\ \frac{c}{24}\sum_{\delta|N} r_\delta/\delta,\ \frac{3(d-1)}{24}\sum_{\delta|N} r_\delta \in \mathbb{Z}$$

and, by virtue of Lemma 2.19,

$$\left(\frac{c}{d}\right)^{2k}\prod_{\delta|N}\left(\frac{\delta}{d}\right)^{r_\delta} = \prod_{\delta|N}\left(\frac{\delta}{d}\right)^{r_\delta} = \prod_{\delta|N}\left(\frac{d}{\delta}\right)^{r_\delta} = \prod_{\delta|N}\left(\frac{1}{\delta}\right)^{r_\delta} = 1.$$

□

Definition 2.21. The almost holomorphic modular form f from Theorem 2.16 will be denoted by $\eta_{1^{r_1}\cdots N^{r_N}}$. Functions of that form are called *eta products for* $\Gamma_1(N)$.

We are mainly interested in holomorphic modular forms and cusp forms. Since $\mathrm{ord}_\infty(\eta) = \frac{1}{24} > 0$, obviously $\eta_{1^{r_1}\cdots N^{r_N}}$ is a modular form if $r_\delta \geq 0$ for all δ and a cusp form if these inequalities are strict. The following gives a precise criterion.

Proposition 2.22. *Let $N > 0$ be an integer and $\eta_{1^{r_1}\cdots N^{r_N}}$ an eta product for $\Gamma_1(N)$. Then*

$$\mathrm{ord}_s(\eta_{1^{r_1}\cdots N^{r_N}}) = \frac{1}{24}\sum_{\delta|N}\frac{(\delta,c)^2 r_\delta}{\delta}$$

for each cusp $s = a/c \in \mathbb{Q}\cup\{\infty\}$ with relatively prime integers a, c.

Proof. Without loss of generality assume $c \geq 0$. Choose $b, d \in \mathbb{Z}$ such that $M\infty = s$ for $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $d = 1$ if $c = 0$. Using again Proposition 2.20, for each positive divisor δ of N we get some integer s_δ and a matrix $M'_\delta \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{aligned}
\eta_{1^{r_1}\cdots N^{r_N}}|_M(\tau) &= \prod_{\delta|N}(c\tau+d)^{-r_\delta/2}\eta\big(\delta(M\tau)\big)^{r_\delta} \\
&= \prod_{\delta|N}\varepsilon(M'_\delta)^{r_\delta}\,\frac{(\delta,c)^{r_\delta/2}}{\delta^{r_\delta/2}}\eta\left(\frac{\big((\delta,c)\tau+s_\delta\big)(\delta,c)}{\delta}\right)^{r_\delta} \\
&= C\prod_{\delta|N}\eta\left(\frac{(\delta,c)^2}{\delta}\tau+s'_\delta\right)^{r_\delta}
\end{aligned}$$

with some $C \in \mathbb{C}^\times$ and $s'_\delta \in \mathbb{Q}$ independent of $\tau \in \mathbb{H}$. This proves

$$\mathrm{ord}_s(\eta_{1^{r_1}\cdots N^{r_N}}) = \sum_{\delta|N} r_\delta\,\mathrm{ord}_\infty\left(\eta\left(\frac{(\delta,c)^2\,\cdot}{\delta}+s'_\delta\right)\right) = \frac{1}{24}\sum_{\delta|N}\frac{(\delta,c)^2 r_\delta}{\delta}.$$

□

This may seem contradictory to Ligozat's result that the order of vanishing of an eta product $\eta_{1^{r_1}\cdots N^{r_N}}$ for $\Gamma_0(N)$ at a cusp a/c (where $c \mid N$ can be assumed) is given by $\frac{N}{24}\sum_{\delta|N}\frac{(\delta,c)^2 r_\delta}{(c,N/c)c\delta}$ [Lig75, Proposition 3.2.8]. It is, of course, not. This is because one has to define the order differently when considering modular forms as functions on the aforementioned Riemann surface $X(\Gamma_0(N))$ (cf. [DS05, Section 3.2]), whilst here we content ourselves with a more naive definition.

The following proposition indicates how eta products can be used to systematically construct bases and compute dimensions of spaces of modular forms with character.

Proposition 2.23. *Let $N > 1$ be an integer, χ, ψ characters* mod N *of* $\Gamma_1(N)$, *$k, \ell \in \mathbb{Z}$ and $f = \eta_{1^{r_1} \dots N^{r_N}}$ an eta product of weight ℓ and character χ for $\Gamma_1(N)$. Then the map*

$$\mathcal{M}^!_k(\Gamma, \psi) \to \mathcal{M}^!_{k+\ell}(\Gamma, \psi\chi) : g \mapsto fg$$

is an isomorphism. If further $\operatorname{ord}_s(f) = 1/N$ *for all cusps s of* $\Gamma_1(N)$, *it restricts to an isomorphism*

$$\mathcal{M}_k(\Gamma, \psi) \to \mathcal{S}_{k+\ell}(\Gamma, \psi\chi) : g \mapsto fg.$$

Proof. As mentioned at the beginning of the section, η does not vanish anywhere on $\mathbb{H}$, so the same holds for f. Since f also has finite order at each cusp, we get a well-defined inverse homomorphism for the first map via

$$\mathcal{M}^!_{k+\ell}(\Gamma, \psi\chi) \to \mathcal{M}^!_k(\Gamma, \psi) : g \mapsto \frac{g}{f}.$$

Now suppose $\operatorname{ord}_s(f) = 1/N$ for all s and let $g \in \mathcal{S}_{k+\ell}(\Gamma, \psi\chi)$. At every cusp s, g has an expansion in powers of q_N, so that $\operatorname{ord}_s(g) \geq 1/N$ and therefore $\operatorname{ord}_s(g/f) \geq 0$. Hence in this case the image of g under the above inverse homomorphism is holomorphic at the cusps, which proves the second assertion. □

The condition $\operatorname{ord}_s(g) \geq 1/N$ could be further weakened, at least if $\psi\chi$ is trivial. However, this would require the introduction of the so-called *width* of a cusp (and its computation), so this refinement will be spared as it is not truly needed in this thesis. Instead, we have a look at a particularly prominent example of an eta product before moving on to theta series.

Example 2.24. Applying Theorem 2.16 to $N = 1$ and $r_1 = 24$ shows that $\eta_{1^{24}} = \eta^{24}$ is an almost holomorphic modular form of weight 12 and trivial character for $\mathrm{SL}_2(\mathbb{Z})$. Moreover, Proposition 2.22 yields $\operatorname{ord}_\infty(\eta^{24}) = 1$ at the only cusp ∞ of $\mathrm{SL}_2(\mathbb{Z})$, so η^{24} is a cusp form. Now we have $\dim(\mathcal{S}_{12}) = 1$ by Theorem 2.10, i.e. the space $\mathcal{S}_{12}$ is indeed generated by η^{24}, which therefore must be a scalar multiple of the discriminant function Δ introduced on page 25. One can compute that the leading term of the Fourier expansion of Δ is $(2\pi)^{12}q$, so $\Delta = (2\pi)^{12}\eta^{24}$. Then Proposition 2.23 shows anew that the homomorphism $\mathcal{M}_{k-12} \to \mathcal{S}_k : f \mapsto \Delta f$ used in the proof of Corollary 2.11 is an isomorphism (even though not completely without use of the dimension formula).

We return to lattices: for the rest of the section, let L be an even lattice with positive-definite quadratic form Q and basis $(e_1, \dots, e_m)$. Then the series $\sum_{x \in L} \exp\bigl(2\pi i Q(x)\tau\bigr)$ converges absolutely locally uniformly in $\tau \in \mathbb{H}$, as can be proved easily using an estimate of the form $Q(\lambda_1 e_1 + \dots + \lambda_m e_m) \geq C(\lambda_1^2 + \dots + \lambda_m^2)$ for all $\lambda_1, \dots, \lambda_m \in \mathbb{Z}$ and some constant $C > 0$.

Definition 2.25. The *theta function* (or *theta series*) of L is defined by

$$\theta_L : \mathbb{H} \to \mathbb{C} : \tau \mapsto \sum_{x \in L} q^{Q(x)}.$$

We see that the coefficient of q^ℓ in the Fourier expansion of θ_L is the number of elements $x \in L$ of norm $Q(x) = \ell$. By definition clearly we have $\theta_L(\tau+1) = \theta_L(\tau)$ for all $\tau \in \mathbb{H}$. Since Q is positive-definite, θ_L is also holomorphic at ∞. Using Poisson summation one can derive a formula for the transformation of θ_L under the action of S as well.

Theorem 2.26. [Ebe13, Proposition 2.1] *For all $\tau \in \mathbb{H}$*

$$\theta_L\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{m/2} \sqrt{\operatorname{disc}(L)}^{-1} \theta_{L'}(\tau).$$

This gives rise to the hope that θ_L could be a modular form of weight $m/2$ for some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Indeed we have the following.

Theorem 2.27. [Ebe13, Theorem 3.2] *Assume that m is even and let N be the level of L. Then θ_L is a (holomorphic) modular form of weight $m/2$ and character*

$$\chi : \Gamma_0(N) \to \mathbb{C}^\times : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\frac{(-1)^{m/2}\operatorname{disc}(L)}{d}\right)$$

for the group $\Gamma_0(N)$.

Example 2.28. The lattice A_2 from Examples 1.11 and 1.15 is positive-definite of even rank 2, level 3 and discriminant 3. Hence $\theta_{A_2} \in \mathcal{M}_1(\Gamma_0(3), \chi)$, where

$$\chi : \Gamma_0(3) \to \mathbb{C}^\times : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\frac{-3}{d}\right)$$

and

$$\theta_{A_2}(\tau) = \sum_{x_1,x_2\in\mathbb{Z}} q^{x_1^2 - x_1x_2 + x_2^2} = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + \cdots$$

for all $\tau \in \mathbb{H}$. For $d = 1 \bmod 3$, Lemma 2.19 yields

$$\left(\frac{-3}{d}\right) = \begin{cases} \left(\frac{d}{-3}\right) = \left(\frac{1}{3}\right) = 1 & \text{if } d > 0, \\ \left(\frac{-3}{-1}\right)\left(\frac{-d}{-3}\right) = -\left(\frac{-d}{3}\right) = -\left(\frac{-1}{3}\right) = 1 & \text{if } d < 0. \end{cases}$$

Thus $\Gamma_1(N) \subseteq \ker(\chi)$ and in particular $\theta_{A_2} \in \mathcal{M}_1(\Gamma_1(3))$.

To obtain the Fourier expansion of θ_{A_2} at the cusp 0, one can apply Theorem 2.26. The Gram matrix of the dual lattice A_2' is $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{smallmatrix}\right)$, so we get

$$\begin{aligned} \theta_{A_2}|_S(\tau) &= \frac{1}{\tau}\theta_{A_2}\left(-\frac{1}{\tau}\right) = \frac{1}{\tau}\left(\frac{\tau}{i}\right)^{2/2}\sqrt{3}^{-1}\sum_{x\in A_2'} q^{Q(x)} = -\frac{i}{\sqrt{3}}\sum_{x_1,x_2\in\mathbb{Z}} q^{(x_1^2+x_1x_2+x_2^2)/3} \\ &= -\frac{i}{\sqrt{3}}\sum_{y_1,y_2\in\mathbb{Z}} q_3^{y_1^2-y_1y_2+y_2^2} = -\frac{i}{\sqrt{3}}\theta_{A_2}\left(\frac{\tau}{3}\right). \end{aligned}$$

Example 2.29. We can do the same for the positive-definite lattice D_4 from Examples 1.12 and 1.15. It has rank 4, level 2 and discriminant 4, so $\theta_{D_4} \in \mathcal{M}_2(\Gamma_0(2)) = \mathcal{M}_2(\Gamma_1(2))$ and

$$\begin{aligned} \theta_{D_4}(\tau) &= \sum_{x_1,x_2,x_3,x_4\in\mathbb{Z}} q^{x_1^2+x_1x_3+x_2^2-x_2x_3+x_3^2-x_3x_4+x_4^2} \\ &= 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + 96q^6 + 192q^7 + 24q^8 + 312q^9 + \cdots \end{aligned}$$

for all $\tau \in \mathbb{H}$.

Using Theorem 2.26, the inverse Gram matrix from Example 1.12 and a slightly more sophisticated change of summation order than in Example 2.28, one computes

$$\begin{aligned}
\theta_{D_4}|_S(\tau) &= \frac{1}{\tau^2}\theta_{D_4}\left(-\frac{1}{\tau}\right) = \frac{1}{\tau^2}\left(\frac{\tau}{i}\right)^{4/2}\sqrt{4}^{-1}\sum_{x\in D_4'} q^{Q(x)} \\
&= -\frac{1}{2}\sum_{x_1,x_2,x_3,x_4\in\mathbb{Z}} q^{(x_1^2-x_1x_2-2x_1x_3-x_1x_4+x_2^2+2x_2x_3+x_2x_4+2x_3^2+2x_3x_4+x_4^2)/2} \\
&= -\frac{1}{2}\sum_{x_1,x_2,x_3,x_4\in\mathbb{Z}} q_2^{(x_1-x_3)^2+x_2^2+x_3^2+(x_2+x_4)^2-(x_1-x_3)(x_2+x_4)+x_3(x_2+x_4)-x_2(x_2+x_4)} \\
&= -\frac{1}{2}\sum_{y_1,y_2,y_3,y_4\in\mathbb{Z}} q_2^{y_1^2+y_1y_3+y_2^2-y_2y_3+y_3^2-y_3y_4+y_4^2} = -\frac{1}{2}\,\theta_{D_4}\left(\frac{\tau}{2}\right).
\end{aligned}$$

Chapter 3

Vector valued modular forms

At the end of the previous chapter, connections between lattices and elliptic modular forms became visible. This suggests that one theory can be exploited to gain insight into the other. In the following both concepts will be merged to a new theory of higher-dimensional modular forms. Where appearance of some character was allowed in Definition 2.8, the presence of the so-called Weil representation will now be mandatory. The link to discriminant forms and in particular the choice of the Weil representation might seem quite arbitrary at first sight. However, the Weil representation has some distinctive properties with deep consequences, some of which are behind what is described in Chapter 4.

Textbooks on the subject are rare, but introductions similar to the following can be found in [Bru02a, Bun01, Sch13, Dit13], for example.

3.1 The Weil representation of $\mathrm{SL}_2(\mathbb{Z})$

In general, for every single metaplectic group the term *Weil representation* refers to a specific linear representation of this group on an infinite-dimensional Hilbert space. This section only deals with a very special version of this representation where the underlying group is just $\mathrm{SL}_2(\mathbb{Z})$ and the Hilbert space can be taken to be finite-dimensional and, more precisely, realized as the group algebra of some discriminant form. Recall the definition of the group algebra.

Definition 3.1. Let G be a group. The *group algebra* of G is the free complex vector space over G, that is, the set

$$\mathbb{C}[G] = \left\{ \sum_{g\in G} \lambda_g g : \lambda_g \in \mathbb{C} \text{ for all } g, \lambda_g \neq 0 \text{ only for finitely many } g \right\}$$

with component-wise addition and scalar multiplication. With multiplication and inner product defined by

$$\left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{h\in G} \mu_h h\right) = \sum_{g,h\in G} \lambda_g \mu_h gh,$$
$$\left\langle \sum_{g\in G} \lambda_g g, \sum_{h\in G} \mu_h h \right\rangle = \sum_{g\in G} \lambda_g \overline{\mu}_g$$

it becomes an algebra and a unitary vector space.

For the rest of the chapter let (D, q) be a discriminant form of level N and even signature $\operatorname{sign}(D)$. Theorem 1.27 shows that this condition is always fulfilled if $4 \nmid N$ since the level of every odd 2-adic Jordan component is divisible by 4. The canonical basis of $\mathbb{C}[D]$ will also be denoted by $(e_\gamma)_{\gamma\in D}$ to reduce confusion, i.e. we write $\sum_{\gamma\in D}\lambda_\gamma e_\gamma$ instead of $\sum_{\gamma\in D}\lambda_\gamma\gamma$ for an element of $\mathbb{C}[D]$.

Definition 3.2. The *Weil representation* ρ_D of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ is defined by

$$\begin{aligned}\rho_D(T)e_\gamma &= e(-\gamma^2/2)e_\gamma,\\ \rho_D(S)e_\gamma &= \frac{e(\operatorname{sign}(D)/8)}{\sqrt{|D|}}\sum_{\beta\in D} e(\gamma\beta)e_\beta\end{aligned}$$

for all $\gamma \in D$. The *dual Weil representation* $\overline{\rho}_D : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[D])$ arises from the usual one via complex conjugation of the corresponding matrices. In particular, for all $\gamma \in D$

$$\begin{aligned}\overline{\rho}_D(T)e_\gamma &= e(\gamma^2/2)e_\gamma,\\ \overline{\rho}_D(S)e_\gamma &= \frac{e(-\operatorname{sign}(D)/8)}{\sqrt{|D|}}\sum_{\beta\in D} e(-\gamma\beta)e_\beta.\end{aligned}$$

Some comments are in order. First, it is not obvious that these representations are well-defined. This issue will be explained in greater detail below. Secondly, ρ_D is a *unitary* representation: for all $\beta, \gamma \in D$ we have

$$\big\langle \rho_D(T)e_\beta, \rho_D(T)e_\gamma\big\rangle = e(\gamma^2/2 - \beta^2/2)\langle e_\beta, e_\gamma\rangle = \delta_{\gamma\beta} = \langle e_\beta, e_\gamma\rangle$$

and

$$\begin{aligned}\big\langle \rho_D(S)e_\beta, \rho_D(S)e_\gamma\big\rangle &= \frac{1}{|D|}\sum_{\alpha,\delta\in D} e(\beta\alpha - \gamma\delta)\langle e_\alpha, e_\delta\rangle\\ &= \frac{1}{|D|}\sum_{\alpha\in D} e(\alpha(\beta-\gamma))\\ &= \delta_{\gamma\beta} = \langle e_\beta, e_\gamma\rangle,\end{aligned}$$

where we have used that

$$\chi : D \to \mathbb{C}^\times : \alpha \mapsto e(\alpha(\beta-\gamma))$$

is a character of D and consequently $\sum_{\alpha\in D}\chi(\alpha) = 0$ if $\chi \neq 1$. This argument will show up some more times. Of course the dual Weil representation is unitary as well. Thirdly, note that in the literature the definitions of ρ_D and $\overline{\rho}_D$ are often interchanged, like in [Bor98, Bru02a, Sch13], whereas this exposition rather sticks to the notation of [Sch09, Sch14]. Either way, $\overline{\rho}_D$ can be considered as the Weil representation corresponding to $(D, -q)$.

Now for the well-definedness of ρ_D. As stated in Section 2.1, $\mathrm{SL}_2(\mathbb{Z})$ is generated by S, T, so there is at most one representation that fits the definition. To prove that it exists, we need to show that the relations between S and T, which were mentioned at the same place, carry over to $\rho_D(T)$ and $\rho_D(S)$: for every $\gamma \in D$,

$$\begin{aligned}
\rho_D(S)^2 e_\gamma &= \frac{e(\operatorname{sign}(D)/4)}{|D|} \sum_{\alpha,\beta\in D} e(\gamma\beta + \beta\alpha) e_\alpha \\
&= \frac{e(\operatorname{sign}(D)/4)}{|D|} \sum_{\alpha\in D} \left(\sum_{\beta\in D} e(\beta(\gamma+\alpha)) \right) e_\alpha \\
&= e(\operatorname{sign}(D)/4) \sum_{\alpha\in D} \delta_{\alpha,-\gamma}\, e_\alpha = e(\operatorname{sign}(D)/4) e_{-\gamma}.
\end{aligned}$$

Since $\operatorname{sign}(D)$ is assumed to be even, this in particular shows that $\rho_D(S)^4 = 1$ as desired. For the relation $S^2(ST)^3$ one can begin similarly,

$$\begin{aligned}
\big(\rho_D(S)\rho_D(T)\big)^3 e_\gamma &= \frac{e(3\operatorname{sign}(D)/8)}{\sqrt{|D|}^3} \sum_{\alpha,\beta,\delta\in D} e\big(-\gamma^2/2 + \gamma\delta - \delta^2/2 + \delta\beta - \beta^2/2 + \beta\alpha\big) e_\alpha \\
&= \frac{e(3\operatorname{sign}(D)/8)}{\sqrt{|D|}^3} \sum_{\alpha,\beta,\delta\in D} e\big(-\gamma^2/2 - \gamma\delta - (\beta+\delta)^2/2 + \beta\alpha\big) e_\alpha \\
&= \frac{e(3\operatorname{sign}(D)/8)}{\sqrt{|D|}^3} \sum_{\alpha,\beta,\delta\in D} e\big(-\gamma^2/2 - \gamma(\delta-\beta) - \delta^2/2 + \beta\alpha\big) e_\alpha \\
&= \frac{e(3\operatorname{sign}(D)/8)}{\sqrt{|D|}^3} \sum_{\alpha,\delta\in D} e\big(-(\gamma+\delta)^2/2\big) \left(\sum_{\beta\in D} e\big(\beta(\alpha+\gamma)\big) \right) e_\alpha \\
&= \frac{e(3\operatorname{sign}(D)/8)}{\sqrt{|D|}} \left(\sum_{\delta\in D} e(-\delta^2/2) \right) e_{-\gamma},
\end{aligned}$$

but the term $\big(\sum_{\delta\in D} e(-\delta^2/2)\big)$ is hard to evaluate. This is an instance of a *Gauss sum*. These objects emerge at various occasions in number theory and some more general Gauss sums are also behind other results cited here (cf. [Sch09, Section 3]). However, for the computation of this one the reader shall be referred to the proof of *Milgram's formula* in [MH73, Appendix 4], which says

$$\left(\sum_{\delta\in D} e(\delta^2/2) \right) = e(\operatorname{sign}(D)/8)\sqrt{|D|}.$$

This finally confirms $\big(\rho_D(S)\rho_D(T)\big)^3 = \rho_D(S)^2$ and therewith the existence of ρ_D.

From here it should be possible, though somewhat demanding, to find an explicit formula for the action of an arbitrary element of $\mathrm{SL}_2(\mathbb{Z})$. This was done in [Sch09, Theorem 4.7].

Theorem 3.3. *Let $\gamma \in D$, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then with notation as in Section 1.4 we have*

$$\rho_D(M) e_\gamma = \xi(M) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta\in D^{c*}} e(-a\beta_c^2/2 - b\beta\gamma - bd\gamma^2/2) e_{d\gamma+\beta}$$

where the root of unity $\xi(M)$ is defined as follows: fix a Jordan decomposition of D. For every prime number p let

$$\xi_p(M) = \begin{cases} \prod_{p|q} \gamma_p\big(q^{\varepsilon_q m_q}\big)\left(\frac{-c}{q^{m_q}}\right) & \text{if } p \text{ is odd and } p \nmid c, \\ \prod_{p|q}\left(\frac{-a}{q^{m_q}}\right)\cdot \prod_{p|q\nmid c}\gamma_p\big((q/(q,c))^{\varepsilon_q m_q}\big)\left(\frac{ac/(q,c)}{(q/(q,c))^{m_q}}\right) & \text{if } p \text{ is odd and } p \mid c, \\ \prod_{2|q}\gamma_2\big(q^{\varepsilon_q m_q}\big) e\big((c+1)\operatorname{oddity}\big(q^{\varepsilon_q m_q}\big)/8\big)\left(\frac{-c}{q^{m_q}}\right) & \text{if } p = 2 \text{ and } 2\nmid c, \\ \prod_{2|q} e\big(-(a+1)\operatorname{oddity}\big(q^{\varepsilon_q m_q}\big)/8\big)\left(\frac{-a}{q^{m_q}}\right)\cdot\prod_{2|q\nmid c}\gamma_2\big((q/(q,c))^{\varepsilon_q m_q}\big) & \\ \qquad e\big((ac/(q,c)-1)\operatorname{oddity}\big((q/(q,c))^{\varepsilon_q m_q}\big)/8\big)\left(\frac{ac/(q,c)}{(q/(q,c))^{m_q}}\right) & \\ & \text{if } p = 2 \text{ and } 2 \mid c \end{cases}$$

where the products extend over all p-adic Jordan components of D. Then let

$$\xi(M) = e\big(\operatorname{sign}(D)/4\big) \prod_{p \text{ prime}} \xi_p(M).$$

Of course all these products are finite since trivial Jordan components only yield factors 1. The dual Weil representation, obtained by substituting q with $-q$ or directly by complex conjugation of the coefficients, is then given by

$$\overline{\rho}_D(M)e_\gamma = \overline{\xi}(M)\frac{\sqrt{|D_c|}}{\sqrt{|D|}} \sum_{\beta\in D^{c*}} e(a\beta_c^2/2 + b\beta\gamma + bd\gamma^2/2)e_{d\gamma+\beta}.$$

As usual the action of congruence subgroups is of special interest.

Definition 3.4. Let

$$\chi_D : \mathbb{Z} \to \mathbb{C}^\times : a \mapsto \left(\frac{a}{|D|}\right) e\big((a-1)\operatorname{oddity}(D)/8\big)$$

and, for all $\gamma \in D$,

$$\chi_\gamma : \mathbb{Z} \to \mathbb{C}^\times : b \mapsto e(-b\gamma^2/2).$$

Proposition 3.5. *Let $\gamma \in D$. Then χ_D and χ_γ are Dirichlet characters* mod N. *Furthermore, their extensions*

$$\chi_D : \Gamma_0(N) \to \mathbb{C}^\times : \begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto \left(\frac{a}{|D|}\right) e\big((a-1)\operatorname{oddity}(D)/8\big)$$

and

$$\chi_\gamma : \Gamma_1(N) \to \mathbb{C}^\times : \begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto e(-b\gamma^2/2)$$

are characters mod N *of* $\Gamma_0(N)$ *and* $\Gamma_1(N)$, *respectively.*

Proof. The assertions regarding the extended maps are immediate from those about the original ones, where that about $\chi_\gamma : \mathbb{Z} \to \mathbb{C}^\times$ is clear, too.

Now let $a, a' \in \mathbb{Z}$ and write $|D| = \prod_{p \text{ prime}} p^{\lambda_p}$. By Proposition 1.9 and Theorem 1.13, $\lambda_p > 0$ if and only if $p \mid N$. First suppose that N is odd. Then D has no 2-adic Jordan components, so that $\operatorname{oddity}(D) = 0 \bmod 8$ and

$$\chi_D(aa') = \left(\frac{aa'}{|D|}\right) = \left(\frac{a}{|D|}\right)\left(\frac{a'}{|D|}\right) = \chi_D(a)\chi_D(a')$$

and

$$\chi_D(a+N) = \prod_{\substack{p \text{ prime} \\ p\neq 2}} \left(\frac{a+N}{p}\right)^{\lambda_p} = \prod_{\substack{p \text{ prime} \\ p\neq 2}} \left(\frac{a}{p}\right)^{\lambda_p} = \chi_D(a).$$

Now suppose that N is even. If also a is even, the above identities are trivial. So assume that a, a' are odd and note that $\operatorname{oddity}(D) = \operatorname{sign}(D) = 0 \bmod 2$ by Theorem 1.27. Thus

$$\begin{aligned}
\chi_D(aa') &= \left(\frac{aa'}{|D|}\right) e\big((a+a'-2)\operatorname{oddity}(D)/8\big)\, e\big((a-1)(a'-1)\operatorname{oddity}(D)/8\big) \\
&= \left(\frac{a}{|D|}\right)\left(\frac{a'}{|D|}\right) e\big((a-1)\operatorname{oddity}(D)/8\big)\, e\big((a'-1)\operatorname{oddity}(D)/8\big) \\
&= \chi_D(a)\chi_D(a').
\end{aligned}$$

If further $8 \mid N$, we have

$$\begin{aligned}
\chi_D(a+N) &= \left(\prod_{p \text{ prime}} \left(\frac{a+N}{p}\right)^{\lambda_p}\right) e\big((a-1)\operatorname{oddity}(D)/8\big)\, e\big(N\operatorname{oddity}(D)/8\big) \\
&= \left(\prod_{p \text{ prime}} \left(\frac{a}{p}\right)^{\lambda_p}\right) e\big((a-1)\operatorname{oddity}(D)/8\big) = \chi_D(a).
\end{aligned}$$

If $N = 2 \bmod 4$, the only 2-adic Jordan components of D are even of exponent 2, so that $\operatorname{oddity}(D) = 0 \bmod 4$ and λ_2 is even. Hence

$$\begin{aligned}
\chi_D(a+N) &= \left(\prod_{\substack{p \text{ prime} \\ p\neq 2}} \left(\frac{a+N}{p}\right)^{\lambda_p}\right) e\big((a+N-1)\operatorname{oddity}(D)/8\big) \\
&= \prod_{\substack{p \text{ prime} \\ p\neq 2}} \left(\frac{a}{p}\right)^{\lambda_p} = \chi_D(a).
\end{aligned}$$

Finally, if $N = 4 \bmod 8$, all odd 2-adic Jordan components of D have exponent 2. Then $\lambda_2 = \operatorname{oddity}(D) = 0 \bmod 2$ and

$$\begin{aligned}
\chi_D(a+N) &= \left(\prod_{\substack{p \text{ prime} \\ p\neq 2}} \left(\frac{a+N}{p}\right)^{\lambda_p}\right) e\big((a-1)\operatorname{oddity}(D)/8\big)\, e\big(N\operatorname{oddity}(D)/8\big) \\
&= \left(\prod_{\substack{p \text{ prime} \\ p\neq 2}} \left(\frac{a}{p}\right)^{\lambda_p}\right) e\big((a-1)\operatorname{oddity}(D)/8\big) = \chi_D(a),
\end{aligned}$$

which completes the proof. $\square$

Now one can easily derive the following from Theorem 3.3, which was done for the case $4 \nmid N$ in [Sch13].

Corollary 3.6. *Let $\gamma \in D$.*

(1) For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have $\rho_D(M)e_\gamma = \chi_D(M)e(-bd\gamma^2/2)e_{d\gamma}$.

(2) For $M \in \Gamma_1(N)$ we have $\rho_D(M)e_\gamma = \chi_\gamma(M)e_\gamma$.

(3) For $M \in \Gamma(N)$ we have $\rho_D(M)e_\gamma = e_\gamma$.

Proof. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then $D_c = D$,

$$\begin{aligned} D^{c*} &= \{\alpha \in D : c\mu^2/2 + \alpha\mu = 0 \bmod 1 \text{ for all } \mu \in D_c\} \\ &= \{\alpha \in D : \alpha\mu = 0 \bmod 1 \text{ for all } \mu \in D\} \\ &= D^\perp = \{0\} = D^c \end{aligned}$$

and, as a trivial consequence, $\beta_c^2/2 = c\mu^2/2 = 0$ for every $\beta \in D^{c*}$, $\mu \in D$. Next, suppose p is a prime number such that D has a p-adic Jordan component $q^{\varepsilon_q m_q}$. Then $q \mid N$ and therefore also $p, q \mid c$. Thus

$$\xi_p(M) = \prod_{p|q} \left(\frac{-a}{q^{m_q}}\right)$$

if p is odd and

$$\begin{aligned} \xi_p(M) &= \prod_{2|q} e\big(-(a+1)\,\mathrm{oddity}\big(q^{\varepsilon_q m_q}\big)/8\big) \left(\frac{-a}{q^{m_q}}\right) \\ &= e\big(-(a+1)\,\mathrm{oddity}(D)/8\big) \prod_{2|q} \left(\frac{-a}{q^{m_q}}\right) \end{aligned}$$

otherwise. Altogether this yields

$$\begin{aligned} \rho_D(M)e_\gamma &= e\big(\mathrm{sign}(D)/4\big) \left(\prod_{p \text{ prime}} \xi_p(M)\right) \frac{\sqrt{|D_c|}}{\sqrt{|D|}} e(-bd\gamma^2/2)e_{d\gamma} \\ &= e\big(\mathrm{sign}(D)/4\big) \left(\frac{-1}{|D|}\right) \chi_D(M)e(-bd\gamma^2/2)e_{d\gamma}. \end{aligned}$$

Inserting $T \in \Gamma_0(N)$ now shows

$$e(-\gamma^2/2)e_\gamma = \rho_D(T)e_\gamma = e\big(\mathrm{sign}(D)/4\big) \left(\frac{-1}{|D|}\right) e(-\gamma^2/2)e_\gamma,$$

which proves claim (1).

Moreover, Proposition 3.5 implies that $\chi_D(a) = \chi_D(1) = 1$ if $a = 1 \bmod N$. Hence for $M \in \Gamma_1(N)$ we get

$$\rho_D(M)e_\gamma = e(-bd\gamma^2/2)e_{d\gamma} = e(-b\gamma^2/2)e_\gamma,$$

which shows (2).

As χ_γ is a character mod N, (3) is now clear. □

Note that this is not the historical order in which these formulae were proved. In fact, the proof of Theorem 3.3 already uses that $\Gamma(N)$ acts trivially and the formula for $\Gamma_0(N)$ appears there as an intermediate step [Sch09, Proposition 4.5].

3.2 Modular forms for ρ_D

In this section the concept of modular forms taking values in the group algebra of some discriminant form will be introduced. The approach to the vector valued case is widely analogous to the definition of classical, scalar valued modular forms that was presented in Chapter 2. However, vector valued modular forms are not just defined as functions whose components are modular forms for some congruence subgroup. Instead they will always be considered as for the full modular group and without additional character, but their different components will be linked via the Weil representation. Hereby congruence subgroups and characters will find their way into the theory all the same.

Let $F : \mathbb{H} \to \mathbb{C}[D]$ be holomorphic on $\mathbb{H}$, i.e. $F = \sum_{\gamma \in D} F_\gamma e_\gamma$ with holomorphic functions $F_\gamma : \mathbb{H} \to \mathbb{C}$, and suppose $F(T\tau) = \rho_D(T)F(\tau)$ for all $\tau \in \mathbb{H}$. Then $e(\gamma^2/2)F_\gamma(\tau + 1) = F_\gamma(\tau)$ for all $\gamma \in D$, that is, the holomorphic function $\mathbb{H} \to \mathbb{C} : \tau \mapsto e(\tau\gamma^2/2)F_\gamma(\tau)$ is 1-periodic and therefore has a q-expansion at ∞. For F_γ itself we get

$$F_\gamma(\tau) = \sum_{m=-\gamma^2/2 \bmod 1} c_\gamma(m)q^m$$

with uniquely determined coefficients $c_\gamma(m) \in \mathbb{C}$ for $m \in \mathbb{Q}$ satisfying $m = -\gamma^2/2 \bmod 1$. Analogously, if $F(T\tau) = \overline{\rho}_D(T)F(\tau)$ for all $\tau \in \mathbb{H}$, every component F_γ has an expansion where the coefficients of powers q^m vanish except when $m = \gamma^2/2 \bmod 1$.

Definition 3.7. Let $k \in \mathbb{Z}$ and $F : \mathbb{H} \to \mathbb{C}[D]$ be holomorphic on $\mathbb{H}$. Then F is called an *(almost holomorphic) modular form for ρ_D of weight k* if

$$j(M, \tau)^{-k}F(M\tau) = \rho_D(M)F(\tau)$$

for all $M \in \mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathbb{H}$ and F is *meromorphic at* ∞, i.e. in the Fourier expansion $F_\gamma(\tau) = \sum_{m=-\gamma^2/2 \bmod 1} c_\gamma(m)q^m$ of each of its components F_γ, all but finitely many coefficients $c_\gamma(m)$ with $m < 0$ vanish. If F is *holomorphic* or *zero at* ∞, i.e. $c_\gamma(m) = 0$ for every $m < 0$ or for every $m \leq 0$, it is a *holomorphic modular form* or a *cusp form for ρ_D of weight k*, respectively. In any case the function

$$\mathbb{H} \to \mathbb{C}[D] : \tau \mapsto \sum_{\gamma \in D} \sum_{\substack{m=-\gamma^2/2 \bmod 1 \\ m<0}} c_\gamma(m)q^m e_\gamma$$

is called the *principal part* of F.

This already implies the definition of modular forms for $\overline{\rho}_D$, namely by applying this definition to $(D, -q)$ instead of (D, q). Therefore most statements in the following will only be made explicitly either for the Weil representation or its dual.

The spaces of almost holomorphic modular forms, holomorphic modular forms and cusp forms for ρ_D of weight k will be denoted by $\mathcal{M}^!_{k,\rho_D}$, $\mathcal{M}_{k,\rho_D}$ and $\mathcal{S}_{k,\rho_D}$, respectively. Note that with this nomenclature a modular form for ρ_D does not have to be holomorphic at ∞, whereas in the scalar valued setting we allow ourselves to simply talk about modular forms only when referring to holomorphic ones. In particular, if D is unimodular, a modular form for ρ_D is an almost holomorphic modular form, but not necessarily a modular form for $\mathrm{SL}_2(\mathbb{Z})$. This convention may not be used homogeneously throughout the literature, but it should be as consistent as possible with [DS05] and [Sch14] at the same time. It is also convenient as we will scarcely be concerned with non-holomorphic elliptic modular forms and with holomorphic modular forms for ρ_D or $\overline{\rho}_D$ that are not cusp forms.

We have the following connection to the spaces $\mathcal{M}_k^!(\Gamma, \chi)$ from the previous chapter.

Proposition 3.8. [Sch13, Theorem 3.2.3] *Let $k \in \mathbb{Z}$, $F = \sum_{\gamma \in D} F_\gamma e_\gamma \in \mathcal{M}^!_{k,\rho_D}$. Then $F_0 \in \mathcal{M}_k^!(\Gamma_0(N), \chi_D)$ and $F_\gamma \in \mathcal{M}_k^!(\Gamma_1(N), \chi_\gamma)$ for all $\gamma \in D$. If $F \in \mathcal{M}_{k,\rho_D}$ or $F \in \mathcal{S}_{k,\rho_D}$, its components are also holomorphic modular forms or cusp forms, respectively.*

Proof. For all $M \in \mathrm{SL}_2(\mathbb{Z})$, $\tau \in \mathbb{H}$ we have

$$\begin{aligned}\sum_{\gamma \in D} F_\gamma|_M(\tau) e_\gamma &= \sum_{\gamma\in D} j(M,\tau)^{-k} F_\gamma(M\tau) e_\gamma = j(M,\tau)^{-k} F(M\tau) = \rho_D(M) F(\tau) \\ &= \sum_{\gamma \in D} F_\gamma(\tau) \rho_D(M) e_\gamma.\end{aligned}$$

Now the asserted transformation behaviour for $M \in \Gamma_0(N)$ or $M \in \Gamma_1(N)$ follows from Corollary 3.6. The computation simultaneously shows that for every $M \in \mathrm{SL}_2(\mathbb{Z})$ the $F_\gamma|_M$ are linear combinations of the F_γ. Thus for any cusp $s \in \mathbb{Q} \cup \{\infty\}$ and $\gamma \in D$, $\mathrm{ord}_s(F_\gamma) \geq \min\big(\{\mathrm{ord}_\infty(F_\gamma) : \gamma \in D\}\big)$, which completes the proof. □

This also motivates some of the constructions done in Section 3.3. Applying Theorem 2.12 to the spaces $\mathcal{M}_k(\Gamma_1(N), \chi_\gamma)$ we further get the following corollary.

Corollary 3.9. *Let $k \in \mathbb{Z}$, $k < 0$, and $F, G \in \mathcal{M}^!_{k,\rho_D}$. If F and G have the same principal part, they are already identical. In particular $\mathcal{M}_{k,\rho_D} = \{0\}$.*

Having therewith confirmed the uniqueness, the next question to ask is when an element of $\mathcal{M}^!_{k,\rho_D}$ with given principal part exists. The following criterion is a key instrument for the classification of automorphic products.

Definition 3.10. Let $F = \sum_{\gamma \in D} F_\gamma e_\gamma$, $G = \sum_{\gamma\in D} G_\gamma e_\gamma$ be any two functions on $\mathbb{H}$ with values in $\mathbb{C}[D]$. Then we call the map

$$(F, \overline{G}) = \sum_{\gamma \in D} F_\gamma G_\gamma : \mathbb{H} \mapsto \mathbb{C}$$

the *pairing* of F and G.

Theorem 3.11. *Let $k \in \mathbb{Z}$.*

(1) For all $\ell \in \mathbb{Z}$, $F \in \mathcal{M}^!_{\ell-k,\rho_D}$, $G \in \mathcal{M}^!_{k,\overline{\rho}_D}$ the pairing is an almost holomorphic modular form of weight ℓ and trivial character for $\mathrm{SL}_2(\mathbb{Z})$.

(2) For all $F \in \mathcal{M}^!_{2-k,\rho_D}$, $G \in \mathcal{M}^!_{k,\overline{\rho}_D}$ the constant coefficient in the q-expansion of $(F, \overline{G})$ vanishes.

(3) *An element of* $\mathcal{M}^!_{2-k,\rho_D}$ *with principal part* $\tilde{F}$ *exists if and only if the constant coefficient of* $(\tilde{F}, \overline{G})$ *vanishes for all* $G \in \mathcal{S}_{k,\overline{\rho}_D}$.

Proof. Since the Weil representation is unitary, (1) is almost obvious. Then (2) can be shown using the residue theorem on the Riemann surface $\bar{\mathbb{C}}$. For the proof of (3) see [Bor99, Theorem 3.1]. □

There are also dimension formulae for the spaces $\mathcal{S}_{k,\overline{\rho}_D}$ [Bor00, Bru02b] which are a bit more complicated to use than Theorems 2.10 and 2.12. Since we will need explicit systems of generators for these spaces anyhow, they will not be stated here.

Instead there is yet another interesting group action to be defined: the group $O(D)$ of all discriminant form automorphisms of D acts on $\mathbb{C}[D]$ by

$$\left(\sum_{\gamma\in D} a_\gamma e_\gamma\right)^\sigma := \sum_{\gamma\in D} a_\gamma e_{\sigma(\gamma)}$$

for all elements $\sum_{\gamma\in D} a_\gamma e_\gamma \in \mathbb{C}[D]$, $\sigma \in O(D)$. Checking the assertion only for the generators S, T of $\mathrm{SL}_2(\mathbb{Z})$ and e_γ of $\mathbb{C}[D]$, it is easy to see that this action commutes with that of ρ_D on $\mathbb{C}[D]$, i.e. $\big(\rho_D(M)x\big)^\sigma = \rho_D(M)x^\sigma$ for all $M \in \mathrm{SL}_2(\mathbb{Z})$, $\sigma \in O(D)$, $x \in \mathbb{C}[D]$. Then for all $k \in \mathbb{Z}$ it induces other actions of $O(D)$ on $\mathcal{M}^!_{k,\rho_D}$, $\mathcal{M}_{k,\rho_D}$ and $\mathcal{S}_{k,\rho_D}$ in the usual way since

$$\begin{aligned} F^\sigma(M\tau) = F(M\tau)^\sigma &= \big(j(M,\tau)^k \rho_D(M) F(\tau)\big)^\sigma \\ &= j(M,\tau)^k \rho_D(M) F(\tau)^\sigma = j(M,\tau)^k \rho_D(M) F^\sigma(\tau) \end{aligned}$$

for all $F \in \mathcal{M}^!_{k,\rho_D}$, $\sigma \in O(D)$, $M \in \mathrm{SL}_2(\mathbb{Z})$. A modular form $F \in \mathcal{M}^!_{k,\rho_D}$ will be called *invariant under a subset* P *of* $O(D)$ if $F^\sigma = F$ for all $\sigma \in P$.

3.3 Constructions of modular forms for ρ_D

As mentioned before, we will have to construct generators for spaces of cusp forms and other specific instances of modular forms for the Weil representation. Therefore some results on relations between spaces of modular forms for ρ_D and, in particular, to the well-known spaces of scalar valued modular forms are desirable. In view of Proposition 3.8 it is rather natural to try and symmetrize scalar valued modular forms for congruence subgroups in order to obtain modular forms for ρ_D of the same weight.

For the next theorem the level of D does not have to be exactly N.

Theorem 3.12. [Sch15, Theorem 3.1] *Let* $k \in \mathbb{Z}$ *and suppose the level of* D *divides* N.

(1) *Let* $f \in \mathcal{M}^!_k(\Gamma_0(N), \chi_D)$ *and* H *be an isotropic subset of* D *which is mapped onto itself under multiplication by elements of* $(\mathbb{Z}/N\mathbb{Z})^\times$. *Then*

$$F = \sum_{M\in\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{\gamma\in H} f|_M \rho_D(M^{-1}) e_\gamma$$

is a modular form for ρ_D *of weight* k *and invariant under* $\{\sigma \in O(D) : \sigma(H) = H\}$.

(2) *Let* $\gamma \in D$ *and* $f \in \mathcal{M}^!_k(\Gamma_1(N), \chi_\gamma)$*. Then*

$$F = \sum_{M \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} f|_M \rho_D(M^{-1}) e_\gamma$$

is a modular form for ρ_D *of weight* k *and invariant under the stabilizer* $O(D)_\gamma$ *of* γ *in* $O(D)$*.*

(3) *Let* $\gamma \in D$ *and* $f \in \mathcal{M}^!_k(\Gamma(N))$*. Then*

$$F = \sum_{M \in \Gamma(N) \backslash \mathrm{SL}_2(\mathbb{Z})} f|_M \rho_D(M^{-1}) e_\gamma$$

is a modular form for ρ_D *of weight* k *and invariant under* $O(D)_\gamma$*.*

Proof. Since the proofs of (1) and (3) are analogous, only (2) shall be proved here. (Moreover, this is the only case that will indeed be applied in Chapter 5.)

First we have to show that F does not depend on the choice of coset representatives. Thus let $M \in \mathrm{SL}_2(\mathbb{Z})$, $K \in \Gamma_1(N)$. Using Corollary 3.6 we get

$$\begin{aligned} f|_{KM} \rho_D\big((KM)^{-1}\big) e_\gamma &= (f|_K)|_M \rho_D(M^{-1}) \rho_D(K^{-1}) e_\gamma \\ &= \chi_\gamma(K) f|_M \rho_D(M^{-1}) \chi_\gamma(K^{-1}) e_\gamma = f|_M \rho_D(M^{-1}) e_\gamma, \end{aligned}$$

so already every single summand in the definition of F is well-defined.

Next, meromorphy of F at ∞ is clear from the assumptions on f. Invariance under $O(D)_\gamma$ follows just as immediately from the fact that ρ_D commutes with the action of $O(D)$.

Now let $K \in \mathrm{SL}_2(\mathbb{Z})$. Then

$$\begin{aligned} j(K,\tau)^{-k} F(K\tau) &= j(K,\tau)^{-k} \sum_{M \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} f|_M(K\tau) \rho_D(M^{-1}) e_\gamma \\ &= \rho_D(K) \sum_{M \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} (f|_M)|_K(\tau) \rho_D(K^{-1}) \rho_D(M^{-1}) e_\gamma \\ &= \rho_D(K) \sum_{M \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} f|_{MK}(\tau) \rho_D\big((MK)^{-1}\big) e_\gamma \\ &= \rho_D(K) \sum_{M' \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} f|_{M'}(\tau) \rho_D\big(M'^{-1}\big) e_\gamma \\ &= \rho_D(K) F(\tau) \end{aligned}$$

for all $\tau \in \mathbb{H}$, which also proves that F transforms correctly under the action of $\mathrm{SL}_2(\mathbb{Z})$. □

The function defined in Theorem 3.12(2) will be called the *lift of* f *(from* $\Gamma_1(N)$*) on* γ, denoted by $F_{\Gamma_1(N),f,\gamma}$, and analogously for parts (1) and (3). Obviously holomorphic modular forms and cusp forms are lifted to holomorphic modular forms and cusp forms for ρ_D, respectively. A slight modification of the above proof of well-definedness further shows that lifting from $\Gamma_1(N)$ is, up to a factor $[\Gamma_1(N) : \Gamma(N)] = N$, just a special case of lifting from $\Gamma(N)$. In practice, however, it will prove most useful because on the one hand, the index of $\Gamma_1(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is of course smaller than that of $\Gamma(N)$. On the other hand, it has the following crucial property, an analogue to which the lifting from $\Gamma_0(N)$ lacks (cf. [Sch15, p. 30]).

Corollary 3.13. *Let $k \in \mathbb{Z}$ and suppose the level of D divides N. For every character χ of $\Gamma_1(N)$ mod N let B_χ be a basis of $\mathcal{S}_k(\Gamma_1(N), \chi)$. Then the set $\{F_{\Gamma_1(N), f, \gamma} : \gamma \in D, f \in B_{\chi_\gamma}\}$ generates the space $\mathcal{S}_{k,\rho_D}$.*

Proof. By Theorem 3.12 this is a subset of $\mathcal{S}_{k,\rho_D}$. Conversely, let $F = \sum_{\gamma \in D} F_\gamma e_\gamma \in \mathcal{S}_{k,\rho_D}$. Then

$$\begin{aligned} F(\tau) &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)]^{-1} \sum_{M \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} j(M,\tau)^{-k} \rho_D(M)^{-1} F(M\tau) \\ &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)]^{-1} \sum_{M \in \Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{\gamma \in D} F_\gamma|_M(\tau) \rho_D(M^{-1}) e_\gamma \\ &= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(N)]^{-1} \sum_{\gamma \in D} F_{\Gamma_1(N), F_\gamma, \gamma}(\tau) \end{aligned}$$

for all $\tau \in \mathbb{H}$, where by virtue of Proposition 3.8 every F_γ is a linear combination of the elements of B_{χ_γ}. □

Sometimes modular forms for ρ_D can also be induced from smaller, but non-trivial discriminant forms.

Theorem 3.14. [Sch15, Theorem 4.1] *Let H be an isotropic subgroup of D and $D_H = H^\perp/H$. Then D_H is a discriminant form of the same signature as D and $|D_H| = |D|/|H|^2$. Further, let $k \in \mathbb{Z}$ and $F_{D_H} = \sum_{\gamma \in D_H} F_{D_H,\gamma}\, e_\gamma \in \mathcal{M}^!_{k,\rho_{D_H}}$. Then $F = \sum_{\gamma \in H^\perp} F_{D_H,\gamma+H}\, e_\gamma$ is a modular form of weight k for ρ_D.*

Proof. Let (V, Q) be a rational quadratic space and L an even lattice in V such that $D = L'/L$. Denote by $\tilde{L}$ the preimage of $H^\perp$ under the projection $L' \to D$. Since H is a group, $\tilde{L}$ is a lattice and $L \subseteq \tilde{L} \subseteq \tilde{L}' \subseteq L'$. In fact we get

$$D_H = H^\perp/H = (\tilde{L}'/L)/(\tilde{L}/L) = \tilde{L}'/\tilde{L},$$

where $\tilde{L}$ is even because H is isotropic. Hence D_H is a discriminant form with

$$\mathrm{sign}(D_H) = \mathrm{sign}(\tilde{L}) = \mathrm{sign}(V) = \mathrm{sign}(L) = \mathrm{sign}(D).$$

The statement about the cardinality is clear from what was said in Section 1.3.

Again the meromorphy of F at ∞ follows from that of F_{D_H}. It therefore remains to show that F transforms correctly under the generators S and T of $\mathrm{SL}_2(\mathbb{Z})$: for all $\tau \in \mathbb{H}$ we get

$$\begin{aligned} j(T,\tau)^{-k} F(T\tau) &= \sum_{\gamma \in H^\perp} F_{D_H,\gamma+H}(\tau) e\big(-(\gamma+H)^2/2\big) e_\gamma \\ &= \sum_{\gamma \in H^\perp} F_{D_H,\gamma+H}(\tau) e(-\gamma^2/2) e_\gamma = \rho_D(T) F(\tau) \end{aligned}$$

and, using that

$$\sum_{\mu \in H} e(\mu\gamma) = \sum_{\mu \in H} e\big((\mu+H)(\gamma+H)\big) = \begin{cases} |H| & \text{if } \gamma \in H^\perp \\ 0 & \text{if } \gamma \in D \setminus H^\perp \end{cases}$$

according to whether $e(\cdot\,\gamma)$ is the trivial character of H or not,

$$\begin{aligned} j(S,\tau)^{-k}F(S\tau) &= \sum_{\gamma\in H^\perp}\left(\frac{e\big(\mathrm{sign}(D_H)/8\big)}{\sqrt{|D_H|}}\sum_{\alpha\in D_H} e\big(\alpha(\gamma+H)\big)F_{D_H,\alpha}(\tau)\right)e_\gamma \\ &= \frac{e\big(\mathrm{sign}(D)/8\big)}{\sqrt{|D|}}|H|\sum_{\gamma\in H^\perp}\sum_{\alpha\in D_H} e\big(\alpha(\gamma+H)\big)F_{D_H,\alpha}(\tau)e_\gamma \\ &= \frac{e\big(\mathrm{sign}(D)/8\big)}{\sqrt{|D|}}\sum_{\gamma\in D}\sum_{\alpha\in D_H} F_{D_H,\alpha}(\tau)\sum_{\mu\in H} e\big((\alpha+(\mu+H))(\gamma+H)\big)e_\gamma \\ &= \frac{e\big(\mathrm{sign}(D)/8\big)}{\sqrt{|D|}}\sum_{\beta\in H^\perp} F_{D_H,\beta+H}(\tau)\sum_{\gamma\in D} e(\beta\gamma)e_\gamma \\ &= \rho_D(S)F(\tau). \end{aligned}$$

□

Example 3.15. [Bru02a, Subsection 1.2.3] Let $k\in\mathbb{Z}$, $k\geq 3$ and $\Gamma_\infty^+=\{T^m : m\in\mathbb{Z}\}$ (so that $\pm\Gamma_\infty^+$ is the stabilizer of ∞ in $\mathrm{SL}_2(\mathbb{Z})$). Then the series

$$E_{k,\overline{\rho}_D}=\frac{1}{2}\sum_{M\in\Gamma_\infty^+\backslash\,\mathrm{SL}_2(\mathbb{Z})} j(M,\cdot\,)^{-k}\,\overline{\rho}_D(M^{-1})e_0$$

defines a holomorphic modular form of weight k for $\overline{\rho}_D$: defining an Eisenstein series for $\Gamma_1(N)$ of weight k by

$$E_{k,\Gamma_1(N)}=\sum_{K\in\Gamma_\infty^+\backslash\Gamma_1(N)} j(K,\cdot\,)^{-k}$$

and using Corollary 3.6(2) indeed we get

$$\begin{aligned} E_{k,\overline{\rho}_D} &= \frac{1}{2}\sum_{M\in\Gamma_1(N)\backslash\,\mathrm{SL}_2(\mathbb{Z})}\ \sum_{K\in\Gamma_\infty^+\backslash\Gamma_1(N)} j(KM,\cdot\,)^{-k}\overline{\rho}_D\big((KM)^{-1}\big)e_0 \\ &= \frac{1}{2}\sum_{M\in\Gamma_1(N)\backslash\,\mathrm{SL}_2(\mathbb{Z})}\ \sum_{K\in\Gamma_\infty^+\backslash\Gamma_1(N)} j(K,M\cdot\,)^{-k}j(M,\cdot\,)^{-k}\overline{\rho}_D(M)^{-1}\overline{\rho}_D(K)^{-1}e_0 \\ &= \frac{1}{2}\sum_{M\in\Gamma_1(N)\backslash\,\mathrm{SL}_2(\mathbb{Z})} E_{k,\Gamma_1(N)}|_M\,\overline{\rho}_D(M)^{-1}e_0. \end{aligned}$$

One can show that $E_{k,\Gamma_1(N)}\in\mathcal{M}_k(\Gamma_1(N))$ [DS05, Section 4.2] and thereby prove that $E_{k,\overline{\rho}_D}$, as the lift of $\frac{1}{2}E_{k,\Gamma_1(N)}$ from $\Gamma_1(N)$ on 0, belongs to $\mathcal{M}_{k,\overline{\rho}_D}$. We will call $E_{k,\overline{\rho}_D}$ *the* Eisenstein series of weight k for the dual Weil representation, although strictly speaking there is more than one object that deserves this name (e.g. we could have lifted on another isotropic element).

Lastly, one should remark that there is also a notion of vector valued theta functions. In fact these initially were among the principal reasons to study the Weil representation. They will not be needed here, though, so we skip this topic.

Chapter 4

Automorphic Products

This chapter finally expands on the concept of automorphic forms for orthogonal groups. The involved theory is only sketched very roughly, not just because an intensive treatment would presuppose sound knowledge of topics like Grassmannians and the Siegel theta function and go beyond the scope of this thesis. It also turns out that, as we are only interested in the objects defined in Section 4.2, every information that is relevant for Chapter 5 can be expressed equivalently in terms of vector valued modular forms. This chapter could therefore be considered as a justification for our interest in the issue treated in Chapter 5 rather than a detailed introduction to the theory. Something closer to the latter can be found in [Bru02a], [BGHZ08], [Bun01] or the original paper [Bor98], for example.

4.1 Automorphic forms for orthogonal groups

Let (L, Q) be an even lattice of type $(n, 2)$ and B the associated bilinear form. As usual, Q and B can be extended to the real vector space $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. Setting

$$B(X + iY, X' + iY') = B(X, X') - B(Y, Y') + iB(X, Y') + iB(Y, X')$$

for all $X, Y, X', Y' \in V$, they can be further extended to the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ of V. Denote by $\mathbb{P}(V \otimes_{\mathbb{R}} \mathbb{C})$ the corresponding projective space, i.e. the set of equivalence classes of $(V \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\}$, where two vectors are equivalent if they differ only by a scalar factor. For $Z = X + iY$, $X, Y \in V$, write $\overline{Z} = X - iY$ for its complex conjugate and $[Z]$ for its image in projective space (if $Z \neq 0$). One can show that the complex manifold

$$\mathcal{K} = \left\{[Z] \in \mathbb{P}(V \otimes_{\mathbb{R}} \mathbb{C}) : Q(Z) = 0, B(Z, \overline{Z}) < 0\right\}$$

has two connected components that are mapped onto each other by conjugation. Choose one of these, call it $\mathcal{K}^+$ and let

$$\tilde{\mathcal{K}}^+ = \left\{Z \in (V \otimes_{\mathbb{R}} \mathbb{C}) \setminus \{0\} : [Z] \in \mathcal{K}^+\right\}$$

be its preimage under the projection. Then $\tilde{\mathcal{K}}^+$ is invariant under multiplication by non-zero complex numbers.

The action of the orthogonal group $O(L)$ can also be extended to V, $V \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{P}(V \otimes_{\mathbb{R}} \mathbb{C})$. Then $\mathcal{K}$ is invariant under $O(L)$ as a set and

$$O(L)^+ = \left\{ \sigma \in O(L) : \sigma(\tilde{\mathcal{K}}^+) = \tilde{\mathcal{K}}^+ \right\}$$

is a subgroup of $O(L)$. This suggests that, when transferring the definition of modular forms from $\mathbb{H}$ to $\tilde{\mathcal{K}}^+$, subgroups of $O(L)^+$ and scalar multiplication should take over the part previously acted by congruence subgroups.

Definition 4.1. Let L be an even lattice of type $(n, 2)$, $n \geq 3$, Γ a subgroup of $O(L)^+$ of finite index, χ a unitary character of Γ and $k \in \mathbb{Z}$. A meromorphic function $\Psi : \tilde{\mathcal{K}}^+ \to \mathbb{C}$ is called an *automorphic form of weight k for Γ with character χ* if $\Psi(tZ) = t^{-k}\Psi(Z)$ and $\Psi(\sigma(Z)) = \chi(\sigma)\Psi(Z)$ for all $z \in \tilde{\mathcal{K}}^+$, $t \in \mathbb{C}^\times$, $\sigma \in \Gamma$.

Here the lower bound on n could be dropped, but then some additional boundary conditions are in order [BGHZ08, Part 2, Definition 2.21]. One can also show that the abelianization of Γ is finite, so that the unitary character χ necessarily has finite order.

Definition 4.2. Let L be an even lattice of type $(n, 2)$. Then the kernel of the natural homomorphism $O(L)^+ \to O(L'/L)$ is called the *discriminant kernel of $O(L)^+$* and will be denoted by Γ_L.

Since L'/L is finite, so are $O(L'/L)$ and the index of Γ_L in $O(L)^+$. For automorphic forms for the discriminant kernel there is an analogue to Corollary 3.9:

Theorem 4.3. [Bun01, Theorem 3.1.19] *Let L be an even lattice of type $(n, 2)$, $n \geq 3$, χ some unitary character of Γ_L and $k \in \mathbb{Z}$. Suppose that L splits two hyperbolic planes over $\mathbb{Q}$, i.e. there is a sublattice $\tilde{L}$ of L such that the rational quadratic spaces $L \otimes_{\mathbb{Z}} \mathbb{Q}$ and $(\tilde{L} \oplus II_{1,1} \oplus II_{1,1}) \otimes_{\mathbb{Z}} \mathbb{Q}$ are isometric. If $k < (n-2)/2$, there are no nontrivial holomorphic automorphic forms for Γ_L of weight k with character χ.*

This is a generalization of Corollary 3.3 in [Bor95], where the lattice is assumed to split the hyperbolic planes over $\mathbb{Z}$. According to Corollary 1.1.25 in [Bun01] the splitting condition of Theorem 4.3 is always fulfilled if $n \geq 5$. For every even lattice of type $(n, 2)$ the number $(n-2)/2$ will be called the *singular weight*.

According to which structure is preferable at the moment, automorphic forms for orthogonal groups can also be considered as defined on other sets than $\tilde{\mathcal{K}}^+$. For instance, let $\mathrm{Gr}(L)$ be the Grassmannian manifold that parametrizes the 2-dimensional negative-definite subspaces of V. Identifying every element $[X + iY]$ of $\tilde{\mathcal{K}}^+$ with the space spanned by X and Y, $\mathrm{Gr}(L)$ can be endowed with the complex structure of $\tilde{\mathcal{K}}^+$, whence the name of [Bor98].

Another approach is the following. A *cusp* of $\tilde{\mathcal{K}}^+$ is a primitive isotropic element of L. If $z \in L$ is a cusp, choose $z' \in L'$ such that $B(z, z') = 1$ and let $K = L \cap z^\perp \cap z'^\perp$ (where $z^\perp = \{z\}^\perp$, $z'^\perp = \{z'\}^\perp$). The lattice K is *Lorentzian*, i.e. it has type $(n-1, 1)$. Similar to the construction of $\tilde{\mathcal{K}}^+$, let

$$\mathcal{H} = \{X + iY \in (K \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} : Q(Y) < 0\}.$$

There is a biholomorphic map from $\mathcal{H}$ to $\mathcal{K}$ [BGHZ08, Lemma 2.18]. Then $\mathcal{K}^+$ can be identified with its preimage in $\mathcal{H}$, which is called the *generalized upper half plane* $\mathbb{H}_n$.

Example 4.4. When taking

$$L = \mathbb{Z}^3, \quad Q : L \to \mathbb{Z} : (x_1, x_2, x_3)^\mathsf{T} \mapsto x_1^2 - x_2^2 - x_3^2,$$

$z = (1, 1, 0)^\mathsf{T}$ and $z' = (\frac{1}{2}, 0, 0)^\mathsf{T}$, we get $K = \mathbb{Z}(0, 0, 1)^\mathsf{T}$ and

$$\mathcal{H} = \left\{ X + iY \in \mathbb{C} : -Y^2 < 0 \right\},$$

so $\mathbb{H}_1 = \mathbb{H}$ or $\mathbb{H}_1 = -\mathbb{H}$, depending on the choice of $\mathcal{K}^+$.

If Ψ is a holomorphic automorphic form for Γ_L, z is a cusp and $\mathbb{H}_n$ the corresponding upper half plane, Ψ can be regarded as a function on $\mathbb{H}_n$ and, as such, has an analogue to the Fourier expansion of a scalar valued modular form.

4.2 The singular theta correspondence

Let L be an even lattice of type $(n, 2)$ with discriminant form D and assume that n is even and $n > 2$. Then a theorem by Borcherds states that specific modular forms for ρ_D of negative weight can be used to construct automorphic forms for some group containing the discriminant kernel Γ_L. The theorem also shows in which way several properties of the arising automorphic form are determined by the input function. For our purposes only the first, easily understandable parts of the theorem are important, but the product expansions are what makes this theory so promising. Unfortunately, neither the definitions required to understand these nor the construction itself can be explained here satisfactorily.

The *Siegel theta function* θ of L is a generalization of the classical theta functions from Section 2.3. In its simpler form, when two possible additional parameters are set to 0 and therefore neglected, it is defined on $\tilde{\mathcal{K}}^+ \times \mathbb{H}$ and takes values in $\mathbb{C}[D]$. So for every $Z \in \tilde{\mathcal{K}}^+$ and $F \in \mathcal{M}^!_{1-n/2,\rho_D}$ we can consider the pairing $\big(F, \overline{\theta}(Z, \cdot)\big)$ as in Definition 3.10. Assume that $F(\tau) = \sum_{\gamma \in D} \sum_{m = -\gamma^2/2 \bmod 1} c_\gamma(m) q^m e_\gamma$ for all $\tau \in \mathbb{H}$ with $c_\gamma(m) \in \mathbb{Z}$ for $m < 0$ and $c_0(0) \in 2\mathbb{Z}$. Then the integral

$$\Phi(Z, F) = \int_{\mathcal{F}} \big(F(x + iy), \overline{\theta(Z, x + iy)}\big) y \frac{\mathrm{d}x \mathrm{d}y}{y^2}$$

over the standard fundamental domain $\mathcal{F}$ of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ does not exist in general, but can be regularized. The function which is discussed in the following theorem is then obtained as $\Psi(F) = \exp\big(\Phi(\cdot, F)\big)$, taking of course the regularized value for Φ. Recall that the *divisor* of Ψ is the family $\big(\mathrm{ord}_Z(\Psi)\big)_{Z \in \tilde{\mathcal{K}}^+}$. Also note that the subgroup

$$O(L, F)^+ = \left\{ \sigma \in O(L)^+ : F^\sigma = F \right\}$$

contains Γ_L and therefore has finite index in $O(L)^+$.

Theorem 4.5. [Bor98, Theorem 13.3] *Let (L, Q) be an even lattice of type $(n, 2)$ with $n > 2$, n even, B its bilinear form, D its discriminant form and $F \in \mathcal{M}^!_{1-n/2,\rho_D}$. As before denote the Fourier coefficients of F by $c_\gamma(m)$, $\gamma \in D$, $m = -\gamma^2/2 \bmod 1$ and suppose that $c_0(0)$ is even and $c_\gamma(m)$ integral for all $\gamma \in D$, $m < 0$. Then there is a meromorphic function $\Psi(F)$ on $\tilde{\mathcal{K}}^+$ with the following properties:*

(1) $\Psi(F)$ *is an automorphic form of weight* $c_0(0)/2$ *for* $O(L,F)^+$ *with some unitary character of* $O(L,F)^+$.

(2) *The divisor of* $\Psi(F)$ *is given by*

$$\big(\Psi(F)\big) = \sum_{\substack{x\in L' \text{ primitive} \\ Q(x)>0}} \sum_{m=1}^{\infty} c_{m(x+L)}\big(-m^2Q(x)\big)x^{\perp}.$$

(3) *For all* $Z \in \tilde{\mathcal{K}}^+$ *we have*

$$\log|\Psi(F)(Z)| = -\Phi(Z,F)/4 + c_0(0)\big(\log(Y) + \Gamma'(1)/2 + \log(\sqrt{2\pi})\big)/2$$

with Φ *as above and* $Z = X + iY$, $X, Y \in L \otimes_{\mathbb{Z}} \mathbb{R}$.

(4) *For each cusp* $z \in L$, *on a suitable open subset of the corresponding generalized upper half plane* $\Psi(F)$ *has a normally convergent product expansion*

$$\Psi(F)(Z) = Ce\big(B(Z,\rho)\big) \prod_{\substack{x\in K', B(x,Z')<0 \\ \text{for all } Z'\in\text{int}(W)}} \prod_{\substack{\gamma\in D \\ p(\gamma)=x+K}} \big(1 - e\big(B(\gamma,z') + B(x,Z)\big)\big)^{c_\gamma(-Q(x))},$$

where z' *and* K *are as above,* p *a certain projection onto* K'/K, C *some constant and* ρ *the so-called* Weyl vector *to the* Weyl chamber W *to which* Z *belongs.*

The map that takes F to $\Psi(F)$ is called the *singular theta correspondence* or the *theta lift* for short. Because of (4), automorphic forms for orthogonal groups that can be formed by means of the singular theta correspondence are called *automorphic products* or *Borcherds products*.

Example 4.6. Let L be an even unimodular lattice of type $(26, 2)$. Then, with notation as above, $\mathcal{M}^!_{1-n/2,\rho_D} = \mathcal{M}^!_{-12}(\mathrm{SL}_2(\mathbb{Z}))$. Since the discriminant function Δ vanishes nowhere on $\mathbb{H}$ (cf. Example 2.24), we have $F := 1/\Delta \in \mathcal{M}^!_{-12}(\mathrm{SL}_2(\mathbb{Z}))$. Computing the q-expansion of F yields $F(\tau) = q^{-1} + 24 + \dots$ for all $\tau \in \mathbb{H}$, so F has integral principal part and even constant coefficient. Thus F can be lifted to an automorphic product $\Psi(F)$ for $O(L,F)^+$. It is known as Φ_{12}. As the principal part is also nonnegative, by Theorem 4.5(2) Φ_{12} is holomorphic. Computing its weight according to part (1) we see that Φ_{12} has indeed singular weight 12.

One can show that $II_{26,2}$ is the only example of a unimodular lattice that carries a holomorphic automorphic product of singular weight and that Φ_{12} is the only such automorphic product [Sch14, Section 4].

Chapter 5

Computations for small discriminant forms

We are now ready to outline the general proceeding when searching for automorphic products with prescribed properties. By definition automorphic products are special cases of automorphic forms for the discriminant kernel that arise via the singular theta correspondence, which reduces our search to the study of modular forms for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$. In particular, Theorem 4.5 contains simple conditions for a vector valued modular form to yield specific weight and poles. After sketching the algorithm and its objective more cleanly than in the introduction, the main steps will be explained in greater detail. Some more results will be stated that did not fit into the general theory due to restrictive assumptions or their computational nature. The final section describes the output of an implementation for levels 2 and 3.

5.1 Problem and general approach

The aim of the algorithm to be described and some of its steps have already been sketched in the introduction or hinted at in the other chapters. Nonetheless, the problem should be formulated once more, clearly and including all details and premises.

Let $p \in \{2, 3\}$ and $n_p \geq 1$. Then we want to find all even lattices L such that the following holds:

- L has level p and genus $II_{n,2}(D)$, where n is even, $n > 2$ and D has p-rank n_p, i.e. $D = 2_{II}^{\varepsilon_2 n_2}$ or $D = 3^{\varepsilon_3 n_3}$ with $\varepsilon_p \in \{\pm 1\}$. Then L satisfies the conditions of Theorem 4.5.
- L splits a hyperbolic plane over $\mathbb{Z}$, i.e. $L = II_{1,1} \oplus \tilde{L}$ for some even lattice $\tilde{L}$.
- L carries a holomorphic automorphic product of singular weight, i.e. there is a function $F \in \mathcal{M}^!_{1-n/2,\rho_D}$ whose theta lift $\Psi(F)$ has weight $(n-2)/2$ and no singularities on $\tilde{\mathcal{K}}^+$.
- We may also want to check if L splits two hyperbolic planes over $\mathbb{Q}$ so that it makes sense to speak of *singular weight* (cf. Theorem 4.3). This will be done retrospectively.

Note that our aim consists only in finding the lattices, not in finding all such automorphic products, even though the algorithm will eventually produce an example in case it exists. If L carries one holomorphic automorphic product of singular weight, in general there may be more.

The proceeding will be as follows. Throughout this chapter, let $k = 1 + n/2$. Suppose L is a lattice of genus $II_{n,2}(D)$, $D = L'/L = p^{\varepsilon_p n_p}$, such that L carries a holomorphic automorphic product of singular weight $\Psi(F)$, where $F = \sum_{\gamma \in D} F_\gamma \in \mathcal{M}^!_{2-k,\rho_D}$. Although not for every combination of p, n_p, ε_p, n the genus $II_{n,2}(p^{\varepsilon_p n_p})$ is non-empty, for given p, n_p there are still infinitely many possibilities for n. In the first step, all but finitely many can be excluded. Here one of the main results of [Sch14], an upper bound on n which only depends on p and n_p, is used. Refining this argument yields only finitely many possibilities for what we will call the *maximal pole order* of the presumed input function F, that is, for

$$m_\infty = \max\bigl(\{-\operatorname{ord}_\infty(F_\gamma) : \gamma \in D\}\bigr).$$

Here we also need that L splits a hyperbolic plane, which may exclude some of the smallest n. Now fixing also m_∞, the function F can be expressed by the vector $\bigl(c_\gamma(m)\bigr)_{\gamma \in D,\, -m_\infty \leq m < 0} \in \mathbb{C}^{p m_\infty |D|}$ which contains those Fourier coefficients of its principal part that are not forced to be zero by the assumption on m_∞. So far, every argument also works for general prime level p.

Next, to implement the different conditions on the variables $c_\gamma(m)$ we will need access to the discriminant form D and its quadratic and bilinear forms. They can be constructed easily by computing a possible Jordan decomposition of D.

The first such condition comes from obstruction theory: for every element of $\mathcal{M}^!_{2-k,\rho_D}$, regardless of whether or not the theta lift applies to it and what would be the result, the constant coefficient of every pairing with an element of the so-called *obstruction space* $\mathcal{S}_{k,\overline{\rho}_D}$ has to vanish. In terms of Fourier coefficients $c_\gamma(m)$, every pairing yields a homogeneous linear equation. A system of generators for $\mathcal{S}_{k,\overline{\rho}_D}$ can be constructed if a basis of $\mathcal{S}_k(\Gamma(p))$ is known. Computer algebra programs are in general capable of computing a basis of $\mathcal{S}_k(\Gamma(p))$ in the form of q-expansions at ∞. But ours has to meet some additional requirements, which is why this will not do. The basis will therefore be built by forming products of other functions whose behaviour is better known. For all the author currently knows, a general procedure that works for arbitrary level has not yet been established, so this part has to be carried out for every level p separately. It will therefore be done only for $p = 2$ and $p = 3$. Up to here, the technique is similar to that applied to $D = 2^{+2}_{II}$ and worked out by hand in [Eis15].

By Theorem 4.5 the singular weight condition is equivalent to $c_0(0) = n - 2$. This coefficient is not included in the set of variables, but nevertheless can be computed using the Eisenstein series $E_{k,\overline{\rho}_D}$ of weight k for $\overline{\rho}_D$. Since the constant coefficient of the pairing with $E_{k,\overline{\rho}_D}$ has to vanish, too, this links the constant Fourier coefficient of F to those from the principal part. The coefficients of $E_{k,\overline{\rho}_D}$ can be computed explicitly and will show that this pairing condition, which gives us another linear equality, is already enough to determine the weight of $\Psi(F)$.

The holomorphy condition means that the divisor of $\Psi(F)$ has to be nonnegative, i.e. for every primitive $x \in L'$ the coefficient of $x^\perp$ in $\bigl(\Psi(F)\bigr)$ as given in

Theorem 4.5 has to be nonnegative. Obviously this results in a number of linear inequalities that the variables $c_\gamma(m)$ for $m < 0$ have to fulfil.

A program that takes p and n_p as input and then computes all these equations and inequalities was written using SageMath Version 6.7 [Sag16]. Since we want the coefficients $c_\gamma(m)$ to be integers so that $\Psi(F)$ can be defined, the result is an integer linear program whose feasibility is equivalent to the existence of the function F. As we are trying to establish theoretical results, no rounding errors are permitted when solving this problem. Unfortunately, Sage lacks the ability to do this except for very small instances where a naive approach also terminates within acceptable time. The full problem and most partial problems whose results for $|D| < 100$ are listed in Section 5.5 were therefore written to a file and handed over to the solver QSopt_ex [ACDE07] executing at the NEOS server [CMM98, GM97, Dol01].

5.2 Bounds on the signature

We need the following crucial result.

Theorem 5.1. [Sch14, Theorems 5.7 and 5.12] *Let L be an even lattice of prime level p and genus $II_{n,2}(p^{\varepsilon_p n_p})$ with $n > 2$ even. Suppose that L splits a hyperbolic plane and carries a holomorphic automorphic product of singular weight. Then for each $c > 1/\log(\frac{\pi e}{6})$ there is a constant d depending only on c such that*

$$n \leq c n_p \log(p) + d.$$

We can take for example $c = 3.59750\ldots$ and $d = 40.52171\ldots$ (or $d = 33.92899\ldots$ if n_p is assumed to be even).

Here the condition that L splits a hyperbolic plane can as well be omitted since otherwise n satisfies the bound automatically, as the following criterion shows.

Proposition 5.2. [Sch14, Propositions 5.2 and 5.9] *Let L be an even lattice of prime level p and genus $II_{n,2}(p^{\varepsilon_p n_p})$ with $n > 2$ even. Then L splits a hyperbolic plane if and only if*

$$n_p = n \quad \textit{and} \quad \varepsilon_p = \left(\frac{-1}{p}\right)^{n_p/2}$$

or

$$n_p \leq n - 1.$$

We will nevertheless have to assume that L splits a hyperbolic plane in order to apply the next theorem below. So for given p, n_p we already have an upper bound and a nontrivial lower bound on n. Also note that Proposition 5.2 and Theorem 1.23 show that we may identify L with its genus.

The oddity formula implies further restrictions. Let $D = p^{\varepsilon_p n_p}$ and first suppose that n_p is even. As p is not a square, Theorems 1.21, 1.22 and 1.27 show that there is a lattice of genus $II_{n,2}(D)$ if and only if

$$e\big((n-2)/8\big) = e\big(\mathrm{sign}(D)/8\big) = \gamma_p(D) = \begin{cases} e\big(\mathrm{oddity}(2_{II}^{\varepsilon_2 n_2})/8\big) = \varepsilon_p = \varepsilon_p\big(\frac{-1}{p}\big)^{n_p/2} & \text{if } p = 2, \\ \varepsilon_p e\big(-(n_p(p-1))/8\big) = \varepsilon_p\big((-1)^{(1-p)/2}\big)^{n_p/2} = \varepsilon_p\big(\frac{-1}{p}\big)^{n_p/2} & \text{if } p > 2. \end{cases}$$

Hence $n = 2 \bmod 4$, $k = 1 + n/2$ is even and ε_p can be computed easily from p, n_p, n. Define $\xi = \varepsilon_p\left(\frac{-1}{p}\right)^{n_p/2}$ for even n_p.

Now suppose that n_p is odd. Similar reasoning shows that there is a lattice of genus $II_{n,2}(D)$ if and only if $p = 1 \bmod 4$ and

$$\begin{aligned} e\big((n-2)/8\big) &= \varepsilon_p\big((-1)^{n_p}\big)^{(1-p)/4} = \varepsilon_p(-1)^{(1-p)/4} \\ &= \varepsilon_p\big((-1)^{(p+1)/2}\big)^{(1-p)/4} = \varepsilon_p\left(\frac{2}{p}\right) \end{aligned}$$

or $p = 3 \bmod 4$ and

$$\begin{aligned} e\big((n-2)/8\big) &= \varepsilon_p\big((-1)^{n_p}\big)^{(-1-p)/4} e\left(\frac{n_p}{4}\right) = \varepsilon_p(-1)^{(p+1)/4} e\left(\frac{n_p}{4}\right) \\ &= \varepsilon_p\big((-1)^{(p-1)/2}\big)^{(p+1)/4} e\left(\frac{n_p}{4}\right) = \varepsilon_p\left(\frac{2}{p}\right)(-1)^{(n_p-1)/2} e\left(\frac{1}{4}\right). \end{aligned}$$

(Note that for $p = 2$ all 2-adic Jordan components of D must be even, so that odd n_p does not occur in this case.) Hence $n = 2 \bmod 4$ if $p = 1 \bmod 4$ and $n = 0 \bmod 4$ if $p = 3 \bmod 4$. In both cases let $\xi = \varepsilon_p\left(\frac{2}{p}\right)\left(\frac{-1}{p}\right)^{(n_p-1)/2}$.

For those lattices that remain after applying these criteria we can use the following.

Theorem 5.3. [Sch14, Propositions 5.6 and 5.11] *Let L be an even lattice of prime level p and genus $II_{n,2}(p^{\varepsilon_p n_p})$ with $n > 2$ even. Suppose that L splits a hyperbolic plane and carries a holomorphic automorphic product of singular weight $\Psi(F)$. Then with m_∞ as in the previous section,*

$$\left(\frac{k-2}{12}\right)^{k-1} \leq m_\infty^{k-1} \leq \xi \frac{p^{(n_p+2)/2}}{p-1} \frac{k-2}{k} B_k$$

if n_p is even and

$$\left(\frac{k-2}{12}\right)^{k-1} \leq m_\infty^{k-1} \leq 3\xi \frac{p^{(n_p+3)/2}}{p-1} \frac{k-2}{k} \frac{B_{k,\chi}}{p^k}$$

otherwise, where $k = 1+n/2$ as before, B_k is the k-th Bernoulli number, $\chi(j) = \left(\frac{j}{p}\right)$ for all $j \in \mathbb{Z}$ and the generalized Bernoulli numbers *for the Dirichlet character χ are defined by the formal power series identity*

$$\sum_{j=1}^{p} \frac{\chi(j) x \exp(jx)}{\exp(px) - 1} = \sum_{m=0}^{\infty} B_{m,\chi} \frac{x^m}{m!}.$$

Since $m_\infty \in \frac{1}{p}\mathbb{Z}$, this leaves us with only finitely many possibilities for m_∞. In fact, it often turns out that there is no element of $\frac{1}{p}\mathbb{Z}$ at all between the bounds on m_∞, which then allows us to discard the lattice in question. For every remaining combination of n and m_∞, by Corollary 3.9 any hypothetical modular form $F \in \mathcal{M}^!_{2-k,\rho_D}$ is uniquely determined by its principal part

$$\sum_{\gamma \in D} \sum_{\substack{-m_\infty \leq m < 0 \\ pm \in \mathbb{Z}}} c_\gamma(m) q^m e_\gamma.$$

This reduces our search for F to a search in $\mathbb{Z}^{pm_\infty|D|}$. Furthermore, many components of a solution $\big(c_\gamma(m)\big)_{\gamma\in D,\,-m_\infty\le m<0}$ are already known to vanish since $c_\gamma(m)\neq 0$ only if $m=-\gamma^2/2 \bmod 1$.

Of course it will not suffice to just enumerate D in some arbitrary way and then consider it as a pure index set. It should rather be constructed and stored together with its group structure, the quadratic and the bilinear form. In order to do this, first consider indecomposable Jordan components. If $p>2$ and $n_p=1$, D is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ with the quadratic form

$$\mathbb{Z}/p\mathbb{Z}\to\mathbb{Q}/\mathbb{Z} : x \bmod p \mapsto \frac{ax^2}{p} \bmod 1,$$

where $a\in\mathbb{Z}$ such that $\left(\frac{2a}{p}\right)=\varepsilon_p$. For $p=2$ and $n_2=2$, D is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ with the quadratic form

$$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\to\mathbb{Q}/\mathbb{Z} : (x \bmod 2, y \bmod 2)\mapsto (x,y)\begin{pmatrix}0 & \frac12\\ 0 & 0\end{pmatrix}(x,y)^{\mathsf{T}} \bmod 1$$

if $\varepsilon_p=+1$ and

$$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\to\mathbb{Q}/\mathbb{Z} : (x \bmod 2, y \bmod 2)\mapsto (x,y)\begin{pmatrix}\frac12 & \frac12\\ 0 & \frac12\end{pmatrix}(x,y)^{\mathsf{T}} \bmod 1$$

if $\varepsilon_p=-1$. Then in general, decomposing $D=p^{\varepsilon_p n_p}$ into indecomposable Jordan components yields an n_p-dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space with a quadratic form described by a diagonal matrix (or block diagonal matrix if $p=2$). This also determines a matrix for the bilinear form.

5.3 Obstruction theory

Next, let us discuss the pairings with cusp forms in greater detail. By Theorem 3.11, the vector $\big(c_\gamma(m)\big)_{\gamma\in D,\,-m_\infty\le m<0}$ defines an element of $\mathcal{M}^!_{2-k,\rho_D}$ exactly if for every cusp form $G\in\mathcal{S}_{k,\overline{\rho}_D}$ with Fourier expansion

$$G:\mathbb{H}\to\mathbb{C}[D] : \tau\mapsto\sum_{\gamma\in D}\sum_{m=\gamma^2/2 \bmod 1} b_\gamma(m)q^m e_\gamma$$

we have

$$0=\sum_{\gamma\in D}\sum_{m=-\gamma^2/2 \bmod 1} b_\gamma(-m)c_\gamma(m)=\sum_{\gamma\in D}\sum_{\substack{m=-\gamma^2/2 \bmod 1\\ -m_\infty\le m<0}} b_\gamma(-m)c_\gamma(m).$$

Hence for every $G\in\mathcal{S}_{k,\overline{\rho}_D}$ we get a linear equation which $\big(c_\gamma(m)\big)_{\gamma\in D,\,-m_\infty\le m<0}$ must solve. Obviously it suffices to check this for a system of generators of $\mathcal{S}_{k,\overline{\rho}_D}$. We obtain such a system by Corollary 3.13, which reduces the problem to finding a basis of $\mathcal{S}_k(\Gamma_1(p),\chi)$ for each character χ mod N of $\Gamma_1(p)$ and then computing the Fourier coefficients of the corresponding lifts from $\Gamma_1(p)$.

As for the computation of the lifts from $\Gamma_1(p)$, suppose $\gamma\in D$ and $g\in\mathcal{S}_k(\Gamma_1(p),\chi_\gamma)$. To compute $G:=F_{\Gamma_1(p),g,\gamma}$ directly from the definition in Theorem 3.12, we would have to compute the Fourier expansion of $g|_M$ and the action of $\rho_D(M^{-1})$ for every element M of a complete system of representatives for

$\Gamma_1(p)\backslash \mathrm{SL}_2(\mathbb{Z})$. But we can do better: for every cusp s of $\Gamma_1(p)$ the summands $g|_M\rho_D(M^{-1})e_\gamma$ with $M\infty = s$ can be grouped together. Then the outer sum extends only over $\varepsilon_\infty(\Gamma_1(p))$ (instead of $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(p)]$) indices, whereas the inner sums can be evaluated using the following formula.

Theorem 5.4. [Sch15, Theorem 3.7] *Let D be any discriminant form of even signature and level dividing $N \neq 4$. Further let $\gamma \in D$, $k \in \mathbb{Z}$, $f \in \mathcal{M}^!_k(\Gamma_1(N), \chi_\gamma)$ and write*

$$F := F_{\Gamma_1(N),f,\gamma} = \sum_{M\in\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})} f|_M\rho_D(M^{-1})e_\gamma = \sum_{s\in\Gamma_1(N)\backslash(\mathbb{Q}\cup\{\infty\})} F_s$$

with

$$F_s = \sum_{\substack{M\in\Gamma_1(N)\backslash \mathrm{SL}_2(\mathbb{Z})\\ M\infty=s}} f|_M\rho_D(M^{-1})e_\gamma$$

for each cusp s of $\Gamma_1(N)$. Now let $a/c \in \mathbb{Q}\cup\{\infty\}$ with $(a, c) = 1$ be a representative of a cusp s, choose $b, d \in \mathbb{Z}$ such that $M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $j \in \mathbb{Z}$ such that $\gamma^2/2 = -j/N \bmod 1$. Further write $f|_M = g_0 + \cdots + g_{N/(c,N,j)-1}$ with $g_\ell|_T = e\big(\ell(c, N, j)/N\big)g_\ell$ for all $\ell \in \{0, \ldots, N/(c, N, j) - 1\}$ (i.e. g_ℓ consists of the terms of order $\ell(c, N, j)/N \bmod 1$ in the q-expansion of $f|_M$) and for every $\mu \in a\gamma + D^{c*}$ choose $j_\mu \in \mathbb{Z}$ such that $\mu^2/2 = -j_\mu(c, N, j)/N \bmod 1$. Then

$$F_s = \xi(M^{-1})\frac{\sqrt{|D_c|}}{\sqrt{|D|}}\frac{N}{(c, N)}\sum_{\mu\in a\gamma+D^{c*}} e\big(d(\mu - a\gamma)^2_c/2 + b\mu\gamma - ab\gamma^2/2\big)g_{j_\mu} \big(e_\mu + (-1)^k e(\mathrm{sign}(D)/4)e_{-\mu}\big)$$

if $N \neq 2$ and

$$F_s = \xi(M^{-1})\frac{\sqrt{|D_c|}}{\sqrt{|D|}}\frac{N}{(c, N)}\sum_{\mu\in a\gamma+D^{c*}} e\big(d(\mu - a\gamma)^2_c/2 + b\mu\gamma - ab\gamma^2/2\big)g_{j_\mu}e_\mu$$

otherwise, where ξ is defined as in Theorem 3.3.

Formulae for $N = 4$ and for lifts from other subgroups can be found in the same article.

We return to the conventions made before the theorem, i.e. $D = p^{\varepsilon_p n_p}$ with prime level $N = p$ and so on. Then the computation of the F_s simplifies greatly. First, $\xi(M^{-1}) = e\big(\mathrm{sign}(D)/4\big)\xi_p(M^{-1})$, where $\xi_p(M^{-1})$ only contains the factor for one Jordan component. Further,

$$\frac{\sqrt{|D_c|}}{\sqrt{|D|}}\frac{p}{(c, p)} = \begin{cases} p^{1-n_p/2} & \text{if } p \nmid c,\\ 1 & \text{if } p \mid c,\end{cases}$$

$$a\gamma + D^{c*} = \begin{cases} D & \text{if } p \nmid c,\\ \{a\gamma\} & \text{if } p \mid c,\end{cases}$$

$$(\mu - a\gamma)^2_c/2 = \begin{cases} 0 & \text{if } p \nmid c,\\ c^{-1}(\mu - a\gamma)^2/2 & \text{if } p \mid c\end{cases}$$

for all $\mu \in a\gamma + D^{c*}$, where the inverse is taken mod p, and

$$\frac{p}{(c,p,j)} = \begin{cases} 1 & \text{if } p \mid c \text{ and } \gamma \text{ is isotropic,} \\ p & \text{otherwise.} \end{cases}$$

This shows that, in particular for prime level p, it is no problem at all to compute the lift of an element $g \in \mathcal{S}_k(\Gamma_1(p), \overline{\chi}_\gamma)$ to an element $G \in \mathcal{S}_{k,\overline{\rho}_D}$, provided that for certain $M \in \mathrm{SL}_2(\mathbb{Z})$ enough is known about $g|_M$. (Note that we have to take the complex conjugate of the character and of the roots of unity that appear in Theorem 5.4 in order to obtain a cusp form for $\overline{\rho}_D$ instead of ρ_D.)

We still need to find a basis of $\mathcal{S}_k(\Gamma(p))$ whose elements can be lifted to cusp forms for $\overline{\rho}_D$. Obviously, it must satisfy the additional condition that every element also be a cusp form for $\Gamma_1(p)$ with some character $\chi \bmod p$. For computational reasons, the basis has to be of an even more specific form, as we need to know how its elements transform when ∞ is mapped to the different cusps of $\Gamma_1(p)$. Such a basis will now be constructed for $p = 2$ and $p = 3$.

Lemma 5.5. *Let $k \in \mathbb{Z}$.*

(1) If $k = 0 \bmod 4$,

$$\mathcal{M}_k(\Gamma_1(2)) = \mathcal{M}_k \oplus \theta_{D_4}^2 \mathcal{M}_{k-4} \oplus \theta_{D_4}^4 \mathcal{M}_{k-8}.$$

(2) If k is even,

$$\mathcal{S}_k(\Gamma_1(2), \chi_1) = \begin{cases} \eta_{1^{12}} \mathcal{M}_{k-6} \oplus \eta_{1^{12}} \theta_{D_4}^2 \mathcal{M}_{k-10} \oplus \eta_{1^{12}} \theta_{D_4}^4 \mathcal{M}_{k-14} & \text{if } k = 2 \bmod 4, \\ \eta_{1^{12}} \theta_{D_4} \mathcal{M}_{k-8} \oplus \eta_{1^{12}} \theta_{D_4}^3 \mathcal{M}_{k-12} \oplus \eta_{1^{12}} \theta_{D_4}^5 \mathcal{M}_{k-16} & \text{if } k = 0 \bmod 4, \end{cases}$$

$$\mathcal{S}_k(\Gamma_1(2), \chi_0) = \begin{cases} \eta_{1^8 2^8} \mathcal{M}_{k-8} \oplus \eta_{1^8 2^8} \theta_{D_4}^2 \mathcal{M}_{k-12} \oplus \eta_{1^8 2^8} \theta_{D_4}^4 \mathcal{M}_{k-16} & \text{if } k = 0 \bmod 4, \\ \eta_{1^8 2^8} \theta_{D_4} \mathcal{M}_{k-10} \oplus \eta_{1^8 2^8} \theta_{D_4}^3 \mathcal{M}_{k-14} \oplus \eta_{1^8 2^8} \theta_{D_4}^5 \mathcal{M}_{k-18} & \text{if } k = 2 \bmod 4 \end{cases}$$

where

$$\chi_j : \Gamma_1(2) \to \mathbb{C}^\times : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto e\left(\tfrac{jb}{2}\right)$$

for $j \in \{0, 1\}$ as in Theorem 2.13. If k is odd, $\mathcal{S}_k(\Gamma_1(2), \chi_1) = \mathcal{S}_k(\Gamma_1(2), \chi_0) = \{0\}$.

(3) If $k = 0 \bmod 6$,

$$\mathcal{M}_k(\Gamma_1(3)) = \mathcal{M}_k \oplus \theta_{A_2}^2 \mathcal{M}_{k-2} \oplus \eta_{1^{-3}3^9} \theta_{A_2}^3 \mathcal{M}_{k-6} \oplus \eta_{1^{-3}3^9} \theta_{A_2}^5 \mathcal{M}_{k-8}.$$

If $k = 3 \bmod 6$,

$$\mathcal{M}_k(\Gamma_1(3)) = \theta_{A_2}^3 \mathcal{M}_{k-3} \oplus \eta_{1^{-3}3^9} \mathcal{M}_{k-3} \oplus \theta_{A_2}^5 \mathcal{M}_{k-5} \oplus \eta_{1^{-3}3^9} \theta_{A_2}^2 \mathcal{M}_{k-5}.$$

(4) If $k = 1 \bmod 3$*,*

$$\mathcal{S}_k(\Gamma_1(3),\chi_1) = \begin{cases} \eta_{1^8}\mathcal{M}_{k-4} \oplus \eta_{1^8}\theta_{A_2}^2\mathcal{M}_{k-6} \\ \quad \oplus \eta_{1^5 3^9}\theta_{A_2}^3\mathcal{M}_{k-10} \oplus \eta_{1^5 3^9}\theta_{A_2}^5\mathcal{M}_{k-12} & \text{if } k = 4 \bmod 6, \\ \eta_{1^8}\theta_{A_2}^3\mathcal{M}_{k-7} \oplus \eta_{1^5 3^9}\mathcal{M}_{k-7} \\ \quad \oplus \eta_{1^8}\theta_{A_2}^5\mathcal{M}_{k-9} \oplus \eta_{1^5 3^9}\theta_{A_2}^2\mathcal{M}_{k-9} & \text{if } k = 1 \bmod 6. \end{cases}$$

Analogous statements hold for $k \neq 1 \bmod 3$ *since*

$$\mathcal{S}_k(\Gamma_1(3),\chi_1) = \begin{cases} \theta_{A_2}\mathcal{S}_{k-1}(\Gamma_1(3),\chi_1) & \text{if } k = 2 \bmod 3, \\ \theta_{A_2}^2\mathcal{S}_{k-2}(\Gamma_1(3),\chi_1) & \text{if } k = 0 \bmod 3. \end{cases}$$

Moreover,

$$\mathcal{S}_k(\Gamma_1(3),\chi_2) = \eta_{1^{-1}3^3}\mathcal{S}_{k-1}(\Gamma_1(3),\chi_1)$$

and

$$\mathcal{S}_k(\Gamma_1(3),\chi_0) = \eta_{1^{-2}3^6}\mathcal{S}_{k-2}(\Gamma_1(3),\chi_1).$$

Here again

$$\chi_j : \Gamma_1(3) \to \mathbb{C}^\times : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto e\left(\tfrac{jb}{3}\right)$$

for $j \in \{0,1,2\}$ *as in Theorem 2.13.*

Proof. If k is negative, (1) is obvious since all spaces are trivial. Otherwise by Examples 2.28 and 2.29 clearly we have

$$\mathcal{M}_k + \theta_{D_4}^2\mathcal{M}_{k-4} + \theta_{D_4}^4\mathcal{M}_{k-8} \subseteq \mathcal{M}_k(\Gamma_1(2))$$

and by Theorems 2.10 and 2.12

$$\begin{aligned} \dim\big(\mathcal{M}_k(\Gamma_1(2))\big) &= \left\lfloor \tfrac{k}{4} \right\rfloor + \left\lfloor \tfrac{k-4}{4} \right\rfloor + \left\lfloor \tfrac{k-8}{4} \right\rfloor + 3 \\ &= \dim(\mathcal{M}_k) + \dim(\mathcal{M}_{k-4}) + \dim \mathcal{M}_{k-8} \\ &= \dim(\mathcal{M}_k) + \dim(\theta_{D_4}^2\mathcal{M}_{k-4}) + \dim(\theta_{D_4}^4\mathcal{M}_{k-8}) \end{aligned}$$

as $\theta_{D_4} \neq 0$ and $k, k-4, k-8 \neq 2 \bmod 12$ (also note that the formula $\dim(\mathcal{M}_\ell) = \left\lfloor \frac{k}{12} \right\rfloor + 1$ applies to $\ell = -4$ and $\ell = -8$ as well).

Hence it remains to show that the spaces on the right hand side are linearly independent. So let

$$f_k \in \mathcal{M}_k,\ f_{k-4} \in \mathcal{M}_{k-4},\ f_{k-8} \in \mathcal{M}_{k-8}$$

and suppose

$$f_k + \theta_{D_4}^2 f_{k-4} + \theta_{D_4}^4 f_{k-8} = 0.$$

Then

$$\theta_{D_4}^2 f_{k-4} + \theta_{D_4}^4 f_{k-8} \in \mathcal{M}_k$$

and therefore

$$\begin{aligned} \theta_{D_4}(\tau)^2 f_{k-4}(\tau) + \theta_{D_4}(\tau)^4 f_{k-8}(\tau) &= (\theta_{D_4}^2 f_{k-4} + \theta_{D_4}^4 f_{k-8})|_S(\tau) \\ &= \tfrac{1}{4}\theta_{D_4}\left(\tfrac{\tau}{2}\right)^2 f_{k-4}(\tau) + \tfrac{1}{16}\theta_{D_4}\left(\tfrac{\tau}{2}\right)^4 f_{k-8}(\tau), \end{aligned}$$

which implies

$$\begin{aligned}\left(\theta_{D_4}(\tau)^2 - \tfrac{1}{4}\theta_{D_4}\left(\tfrac{\tau}{2}\right)^2\right) f_{k-4}(\tau) &= \left(\tfrac{1}{16}\theta_{D_4}\left(\tfrac{\tau}{2}\right)^4 - \theta_{D_4}(\tau)^2\right) f_{k-8}(\tau)\\ &= \left(\theta_{D_4}(\tau)^2 - \tfrac{1}{4}\theta_{D_4}\left(\tfrac{\tau}{2}\right)^2\right)\left(-\theta_{D_4}(\tau)^2 - \tfrac{1}{4}\theta_{D_4}\left(\tfrac{\tau}{2}\right)^2\right) f_{k-8}(\tau)\end{aligned}$$

for all $\tau \in \mathbb{H}$. Thus

$$\tfrac{1}{4}\theta_{D_4}\left(\tfrac{\cdot}{2}\right)^2 + \theta_{D_4}^2 = -\frac{f_{k-4}}{f_{k-8}} \in \mathcal{M}_4$$

if $f_{k-8} \neq 0$, since the left hand side is holomorphic on $\mathbb{H}$ and at the cusps and the right hand side is modular for $\mathrm{SL}_2(\mathbb{Z})$. This is a contradiction because the Fourier expansion of $\frac{1}{4}\theta_{D_4}\left(\frac{\cdot}{2}\right)^2$ contains half-integral powers of q, so $f_{k-8} = 0$. Then $\left(\theta_{D_4}^2 - \frac{1}{4}\theta_{D_4}\left(\frac{\cdot}{2}\right)^2\right)f_{k-4} = 0$ and the same argument shows that $f_{k-4} = 0$ and consequently also $f_k = 0$, which proves (1).

For odd k and $k \leq 2$, (2) is clear by Theorem 2.12. Otherwise by Theorem 2.16 $\eta_{1^{12}}$ is an almost holomorphic modular form of weight 6 and character χ_1 for $\Gamma_1(2)$. As according to Proposition 2.22 we have $\mathrm{ord}_\infty(\eta_{1^{12}}) = \mathrm{ord}_0(\eta_{1^{12}}) = \frac{1}{2}$, the first statement for $k = 2 \bmod 4$ now follows from (1) and Proposition 2.23.

For $k = 0 \bmod 4$ the same argument shows

$$\begin{aligned}\dim\big(\mathcal{S}_k(\Gamma_1(2), \chi_1)\big) &= \dim\big(\eta_{1^{12}}\mathcal{M}_{k-6}(\Gamma_1(2))\big) = \left\lfloor \tfrac{k-6}{4} \right\rfloor + 1 = \left\lfloor \tfrac{k-8}{4} \right\rfloor + 1\\ &= \dim\big(\eta_{1^{12}}\mathcal{M}_{k-8}(\Gamma_1(2))\big) = \dim\big(\mathcal{S}_{k-2}(\Gamma_1(2), \chi_1)\big)\end{aligned}$$

so that

$$\mathcal{S}_{k-2}(\Gamma_1(2), \chi_1) \to \mathcal{S}_k(\Gamma_1(2), \chi_1) : f \mapsto \theta_{D_4} f$$

is an isomorphism. This proves the first statement also for $k = 0 \bmod 4$.

Next, by Theorem 2.16 and Proposition 2.22 we have $\eta_{1^8 2^8} \in \mathcal{S}_8(\Gamma_1(2)) = \mathcal{S}_8(\Gamma_1(2), \chi_0)$. Now using

$$\dim\big(\mathcal{S}_k(\Gamma_1(2))\big) = \left\lfloor \tfrac{k}{4} \right\rfloor - 1 = \begin{cases} \left\lfloor \tfrac{k-8}{4} \right\rfloor + 1 = \dim\big(\mathcal{M}_{k-8}(\Gamma_1(2))\big) & \text{if } k = 0 \bmod 4,\\ \left\lfloor \tfrac{k-10}{4} \right\rfloor + 1 = \dim\big(\mathcal{M}_{k-10}(\Gamma_1(2))\big) & \text{if } k = 2 \bmod 4\end{cases}$$

and the above isomorphism again, we get the second statement from (1), too.

As for (3), suppose $k = 0 \bmod 6$. Again, for $k < 0$ the whole statement is obvious. Otherwise the inclusion

$$\mathcal{M}_k + \theta_{A_2}^2\mathcal{M}_{k-2} + \eta_{1^{-3}3^9}\theta_{A_2}^3\mathcal{M}_{k-6} + \eta_{1^{-3}3^9}\theta_{A_2}^5\mathcal{M}_{k-8} \subseteq \mathcal{M}_k(\Gamma_1(3))$$

follows easily from Theorem 2.16, Proposition 2.22 and Example 2.28. Analogously to the calculation in (1) we see that

$$\begin{aligned}&\dim\big(\mathcal{M}_k(\Gamma_1(3))\big)\\ &\quad= \dim(\mathcal{M}_k) + \dim(\theta_{A_2}^2\mathcal{M}_{k-2}) + \dim(\eta_{1^{-3}3^9}\theta_{A_2}^3\mathcal{M}_{k-6}) + \dim(\eta_{1^{-3}3^9}\theta_{A_2}^5\mathcal{M}_{k-8}).\end{aligned}$$

Therefore let $f_\ell \in \mathcal{M}_\ell$ for $\ell \in \{k, k-2, k-6, k-8\}$ and assume

$$f_k + \theta_{A_2}^2 f_{k-2} = \eta_{1^{-3}3^9}\theta_{A_2}^3 f_{k-6} + \eta_{1^{-3}3^9}\theta_{A_2}^5 f_{k-8}.$$

If the left hand side does not vanish, it has order $\min\bigl(\mathrm{ord}_\infty(f_k), \mathrm{ord}_\infty(f_{k-2})\bigr)$ both at ∞ and 0 (since θ_{A_2} does not vanish at any cusp), whereas the right hand side by Proposition 2.22 has order $\min\bigl(1 + \mathrm{ord}_\infty(f_{k-6}), 1 + \mathrm{ord}_\infty(f_{k-8})\bigr)$ at ∞ and $\min\bigl(\mathrm{ord}_\infty(f_{k-6}), \mathrm{ord}_\infty(f_{k-8})\bigr)$ at 0. The latter two are not equal, which is a contradiction. So

$$f_k + \theta_{A_2}^2 f_{k-2} = \eta_{1^{-3}3^9}\theta_{A_2}^3(f_{k-6} + \theta_{A_2}^2 f_{k-8}) = 0$$

and as in (1) we get

$$\theta_{A_2}^2 = \frac{f_k}{f_{k-2}} \in \mathcal{M}_2$$

if $f_{k-2} \neq 0$ and

$$\theta_{A_2}^2 = \frac{f_{k-6}}{f_{k-8}} \in \mathcal{M}_2$$

if $f_{k-8} \neq 0$. But $\mathcal{M}_2 = \{0\}$ and consequently

$$f_k = f_{k-2} = f_{k-6} = f_{k-8} = 0.$$

For the case $k = 3 \bmod 6$ and for (4) the proofs are completely analogous to those for $k = 0 \bmod 6$ and (2), respectively. □

For each of the spaces $\mathcal{M}_\ell$ a basis can be found easily, for example using Corollary 2.11. Since by Theorem 2.4 both $\Gamma_1(2)$ and $\Gamma_1(3)$ only have two cusps, namely those represented by $\infty = 1\infty$ and $0 = S\infty$, and the transformation behaviour of θ_{A_2}, θ_{D_4} (cf. Examples 2.28 and 2.29) and eta products under S is well known – for example,

$$\eta_{1^8 2^8}|_S(\tau) = \tau^{-8}\sqrt{-i\tau}^{\,8}\eta(\tau)^8\sqrt{-i\frac{\tau}{2}}^{\,8}\eta\Bigl(\Bigl(\frac{\tau}{2}\Bigr)^8 = \frac{1}{16}\eta_{1^8 2^8}\Bigl(\frac{\tau}{2}\Bigr)$$

for all $\tau \in \mathbb{H}$ –, the lifts of a basis obtained from Lemma 5.5(2) and (4) can be computed.

Further, in order to eventually get equations with rational (or integral) coefficients instead of arbitrary complex ones, it is desirable that the Fourier expansions of the lifts have coefficients in some finite extension K of $\mathbb{Q}$. Every linear equation with coefficients in K can then be split into $[K : \mathbb{Q}]$ equations with rational coefficients. Theorem VI.3.3 in [Lan02] and a careful examination of the factors appearing in Theorem 5.4, the Fourier coefficients of the functions involved in Lemma 5.5 and those of their transformations under S show that we can take $K = \mathbb{Q}$ for $p = 2$ and $K = \mathbb{Q}(\zeta_{12})$ for $p = 3$, where ζ_{12} is a primitive 12-th root of unity.

5.4 Singular weight and holomorphy

As already mentioned, the Eisenstein series $E_{k,\overline{\rho}_D}$ of weight k for the dual Weil representation will be used to determine the coefficient $c_0(0)$ from the principal part of F. Recall its definition from Example 3.15 and denote its Fourier coefficients by $a_\gamma(m)$, i.e.

$$E_{k,\overline{\rho}_D} : \mathbb{H} \to \mathbb{C}[D] : \tau \mapsto \sum_{\gamma \in D} \sum_{\substack{m = \gamma^2/2 \bmod 1 \\ m \geq 0}} a_\gamma(m) q^m e_\gamma.$$

For the original Weil representation the $a_\gamma(m)$ were computed in great generality in [Sch06]. For prime level we get the following, simpler formula.

Proposition 5.6. [Sch06, Theorem 5.1 and Section 7], [Sch14, Propositions 5.1 and 5.8] *Let L be an even lattice of prime level p and genus $II_{n,2}(D)$ with $n > 2$ even, $D = p^{\varepsilon_p n_p}$ and $\gamma \in D$. Define*

$$\sigma_\ell(m) = \sum_{\delta | m} \delta^\ell,$$

$$\sigma_{\ell,\chi}(m) = \sum_{\delta | m} \chi(\tfrac{m}{\delta})\delta^\ell$$

for all integers $\ell \geq 0$, $m > 0$ and

$$c_{k,p,n_p} = \begin{cases} \xi \frac{2k}{B_k} \frac{1}{p^k - 1} \frac{1}{p^{(n_p-2)/2}} & \text{if } n_p \text{ is even,} \\ \xi \frac{2k}{B_{k,\chi}} \frac{1}{p^{(n_p-1)/2}} & \text{if } n_p \text{ is odd,} \end{cases}$$

where again $\chi : \mathbb{Z} \to \mathbb{C}^\times : j \mapsto \left(\frac{j}{p}\right)$. Then

$$a_\gamma(0) = \begin{cases} 1 & \text{if } \gamma = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Further, let $m > 0$, $m = \gamma^2/2 \bmod 1$, and $\nu, a \in \mathbb{Z}$ such that $m = p^\nu a$, $(a,p) = 1$. Then

$$a_\gamma(m) = \begin{cases} -c_{k,p,n_p}\sigma_{k-1}(pm) & \text{if } \gamma^2/2 \neq 0 \bmod 1, \\ -c_{k,p,n_p}p^{(\nu+1)(k-1)}\sigma_{k-1}(a) + \xi c_{k,p,n_p}p^{n_p/2}\sigma_{k-1}(a) & \\ \qquad - \xi c_{k,p,n_p}p^{(n_p-2)/2}(p-1)\sigma_{k-1}(m) & \text{if } \gamma = 0, \\ -c_{k,p,n_p}p^{(\nu+1)(k-1)}\sigma_{k-1}(a) & \text{otherwise} \end{cases}$$

if n_p is even and

$$a_\gamma(m) = \begin{cases} -c_{k,p,n_p}\sigma_{k-1,\chi}(pm) & \text{if } \gamma^2/2 \neq 0 \bmod 1, \\ -c_{k,p,n_p}p^{(\nu+1)(k-1)}\sigma_{k-1,\chi}(a) & \\ \qquad - \xi c_{k,p,n_p}p^{(n_p-1)/2}\chi(a)\sigma_{k-1,\chi}(a) & \text{if } \gamma = 0, \\ -c_{k,p,n_p}p^{(\nu+1)(k-1)}\sigma_{k-1,\chi}(a) & \text{otherwise} \end{cases}$$

if n_p is odd.

Hence by Theorem 3.11(2) and Theorem 4.5(1), $\Psi(F)$ has singular weight if and only if

$$-\sum_{\gamma \in D} \sum_{-m_\infty \leq m < 0} a_\gamma(-m)c_\gamma(m) = \sum_{\gamma \in D} a_\gamma(0)c_\gamma(0) = c_0(0) = n - 2 = 2(k-2).$$

Since by Proposition 5.6 the computation of the $a_\gamma(-m)$ is trivial to implement, this allows the algorithm to deal with the singular weight issue.

As to the holomorphy condition, computing the divisor as stated in Theorem 4.5(2) is not easy. (For example, shortly we will see that there are infinitely many primitive $x \in L'$.) Luckily, also here a more convenient condition is known.

Proposition 5.7. [Sch14, Proposition 5.3] *Let (L, Q) be an even lattice of prime level p and genus $II_{n,2}(p^{\varepsilon_p n_p})$ with $n > 2$ even. Suppose that L splits a hyperbolic plane and $F \in \mathcal{M}^!_{2-k,\rho_D}$ has Fourier coefficients $c_\gamma(m)$ with $c_\gamma(m) \in \mathbb{Z}$ for $m < 0$ and $c_0(0) \in 2\mathbb{Z}$. Let*

$$g_\gamma(d) = \sum_{m=1}^{\infty} c_{m\gamma}(-m^2 d)$$

for all $\gamma \in D$ and positive $d \in \frac{1}{p}\mathbb{Z}$. Then the automorphic product $\Psi(F)$ is holomorphic if and only if $g_\gamma(d) \geq 0$ for all γ, d.

Proof. First assume that $g_\gamma(d) \geq 0$ for all γ, d and let $x \in L'$ be primitive with $Q(x) > 0$. Then the coefficient of the divisor $x^\perp$ in $\big(\Psi(F)\big)$ is given by

$$\sum_{m=1}^{\infty} c_{m(x+L)}\big(-m^2 Q(x)\big) = g_{x+L}\big(Q(x)\big) \geq 0.$$

So $\big(\Psi(F)\big) \geq 0$, i.e. $\Psi(F)$ is holomorphic.

Now conversely assume that $\Psi(F)$ is holomorphic and let $\gamma \in D$, $d \in \frac{1}{p}\mathbb{Z}$, $d > 0$. If $\gamma^2/2 \neq d \bmod 1$, we have

$$g_\gamma(d) = g_0(p^2 d) + \sum_{\substack{1 \leq m < \infty \\ (m,p)=1}} c_{m\gamma}(-m^2 d) = g_0(p^2 d).$$

Therefore we may assume that $\gamma^2/2 = d \bmod 1$. Write $L = \tilde{L} \oplus II_{1,1}$, $L' = \tilde{L}' \oplus II_{1,1}$ and let $x = \tilde{x} + y \in L'$ with $\tilde{x} \in \tilde{L}'$, $y \in II_{1,1}$, $x + L = \gamma$. Since $II_{1,1} \subseteq L$ and $II_{1,1}$ contains primitive elements of arbitrary norm (cf. Example 1.10), we may assume that $Q(x) = Q(\tilde{x}) + Q(y) = d$ and that y is primitive. Then x is primitive, too, and

$$g_\gamma(d) = \sum_{m=1}^{\infty} c_{m\gamma}(-m^2 d) = \sum_{m=1}^{\infty} c_{m(x+L)}\big(-m^2 Q(x)\big) \geq 0$$

as $\big(\Psi(F)\big) \geq 0$. □

This shows that the integer linear program describing the existence of the holomorphic automorphic product $\Psi(F)$ of singular weight can be completed by adding an inequality

$$\sum_{m=1}^{\lfloor (m_\infty/d)^{1/2} \rfloor} c_{m\gamma}(-m^2 d) = \sum_{m=1}^{\infty} c_{m\gamma}(-m^2 d) = g_\gamma(d) \geq 0$$

for every pair $(\gamma, d) \in D \times \frac{1}{p}\mathbb{Z}$ with $0 < d \leq m_\infty$, $d = \gamma^2/2 \bmod 1$. Since this in particular implies that $c_\gamma(-m_\infty) \geq 0$ for all $\gamma \in D$, a single inequality

$$\sum_{\gamma \in D} c_\gamma(-m_\infty) \geq 1$$

can be added to prescribe that the maximal pole order of F be *exactly* m_∞.

5.5 Results

Applying the whole procedure to lattices L with $|L'/L| < 100$, i.e. to 2-ranks $2, 4, 6$ and 3-ranks $1, 2, 3, 4$, yields the results listed on the next pages. The tables are to be read as follows.

The first column contains the genus which is examined. The second gives all possibilities for the maximal pole order of a hypothetical input function $F \in \mathcal{M}^!_{2-k,\rho_D}$ which are not excluded by Theorem 5.3. The third column states whether (y) or not (n) the lattice carries a holomorphic automorphic product $\Psi(F)$ of singular weight such that F has maximal pole order m_∞.

In the remaining columns additional information on different aspects of the problem is displayed. The fourth contains the number of degrees of freedom of the problem (deg. f.), that is, the number of variables $c_\gamma(m)$ that satisfy $m_\infty \leq m < 0$ and $m = -\gamma^2/2 \bmod 1$ (cf. Section 5.2). In the fifth column the number of $\mathbb{Q}$-linearly independent conditions obtained from obstruction theory (obstr.) is given. Note that this need not be the dimension of $\mathcal{S}_{k,\overline{\rho}_D}$ over $\mathbb{C}$ (cf. Section 5.3). Initially also

$$\min\bigl(\{\min_{\gamma \in D} \operatorname{ord}_\infty(G_\gamma) : G \in \mathcal{S}_{k,\overline{\rho}_D}\}\bigr)$$

was computed to check if for small $|m|$ some of the $c_\gamma(m)$ are not touched by the obstruction conditions. However, the results are not included in the table since here in every case the result coincides with the smallest possible value that can be expected from the rest of the data (i.e. ∞ if there are no nontrivial obstruction conditions, $\frac{2}{3}$ for $D = 3^{+1}$ and $\frac{1}{p}$ otherwise).

The other columns make statements about the solvability of different partial problems in order to reveal which are the "crucial" conditions. The first of these states whether there is a rational solution for the single equation given by the pairing with $E_{k,\overline{\rho}_D}$ (cf. Section 5.4). This condition is very weak because it is already fulfilled if there is any pair $(\gamma, m) \in D \times \frac{1}{p}\mathbb{Z}$ such that $0 < m \leq m_\infty$ and $a_\gamma(m) \neq 0$. The next columns do the same for integral solutions, for integral solutions that simultaneously satisfy the inequalities $g_\gamma(d) \geq 0$ from the holomorphy condition (note that $\mathbb{Z}_{\text{hol}}$ is an abbreviation rather than a well-defined subset of $\mathbb{Z}$) and for nonnegative integral solutions. For the last three columns, the equations from obstruction theory are added to the combinations considered before. The naming should be intuitive. A column $\mathbb{Z}_{\text{hol},\mathcal{S}}$ is missing as this would be equivalent to the full problem (without prescribing m_∞), which can already be seen from the third column.

Note that the holomorphy condition also plays a role in the columns $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Q}_\mathcal{S}$ and $\mathbb{Z}_\mathcal{S}$, where the inequalities from Section 5.3 are ignored: it is essential for determining the possibilities for m_∞ by means of Theorem 5.3 and thereby already influences the space $\mathbb{Q}^{pm_\infty|D|}$ in which our search takes place. Besides, without demanding for holomorphy it would not make sense either to restrict the search to those signatures listed in the table.

For levels $p = 2$ and $p = 3$, respectively, we get the following tables.

L	m_∞	Ψ	deg. f.	obstr.	$\mathbb{Q}$	$\mathbb{Z}$	$\mathbb{Z}_{\text{hol}}$	$\mathbb{Z}_{\geq 0}$	$\mathbb{Q}_{\mathcal{S}}$	$\mathbb{Z}_{\mathcal{S}}$	$\mathbb{Z}_{\geq 0,\mathcal{S}}$
$II_{10,2}(2_{II}^{+2})$	$\frac{1}{2}$	y	1	0	y	y	y	y	y	y	y
$II_{14,2}(2_{II}^{-2})$	$\frac{1}{2}$	n	3	1	y	n	n	n	n	n	n
$II_{26,2}(2_{II}^{+2})$	$\frac{2}{2}$	y	4	2	y	y	y	y	y	y	y
$II_{6,2}(2_{II}^{-4})$	$\frac{1}{2}$	n	10	0	y	n	n	n	y	n	n
$II_{10,2}(2_{II}^{+4})$	$\frac{1}{2}$	y	6	1	y	y	y	y	y	y	y
$II_{14,2}(2_{II}^{-4})$	$\frac{1}{2}$	n	10	5	y	n	n	n	n	n	n
$II_{22,2}(2_{II}^{-4})$	$\frac{2}{2}$	n	16	11	y	n	n	n	y	n	n
$II_{26,2}(2_{II}^{+4})$	$\frac{2}{2}$	y	16	11	y	y	y	y	y	y	y
$II_{38,2}(2_{II}^{-4})$	$\frac{3}{2}$	n	26	21	y	n	n	n	n	n	n
$II_{10,2}(2_{II}^{+6})$	$\frac{1}{2}$	y	28	7	y	y	y	y	y	y	y
$II_{14,2}(2_{II}^{-6})$	$\frac{1}{2}$	n	36	21	y	n	n	n	n	n	n
$II_{22,2}(2_{II}^{-6})$	$\frac{2}{2}$	n	64	43	y	y	y	y	y	n	n
$II_{26,2}(2_{II}^{+6})$	$\frac{2}{2}$	y	64	49	y	y	y	y	y	y	y
$II_{38,2}(2_{II}^{-6})$	$\frac{3}{2}$	n	100	85	y	y	y	y	n	n	n

L	m_∞	Ψ	deg. f.	obstr.	$\mathbb{Q}$	$\mathbb{Z}$	$\mathbb{Z}_{\text{hol}}$	$\mathbb{Z}_{\geq 0}$	$\mathbb{Q}_{\mathcal{S}}$	$\mathbb{Z}_{\mathcal{S}}$	$\mathbb{Z}_{\geq 0,\mathcal{S}}$
$II_{4,2}(3^{-1})$	$\frac{1}{3}$	n	2	0	y	n	n	n	y	n	n
$II_{8,2}(3^{+1})$	$\frac{1}{3}$	n	0	0	n	n	n	n	n	n	n
$II_{12,2}(3^{-1})$	$\frac{2}{3}$	n	2	1	y	n	n	n	n	n	n
$II_{16,2}(3^{+1})$	$\frac{2}{3}$	n	2	1	y	n	n	n	n	n	n
$II_{24,2}(3^{+1})$	$\frac{3}{3}$	n	3	1	y	n	n	n	y	n	n
$II_{32,2}(3^{+1})$	$\frac{4}{3}$	n	3	2	y	n	n	n	n	n	n
$II_{6,2}(3^{+2})$	$\frac{1}{3}$	n	4	1	y	n	n	n	y	n	n
$II_{10,2}(3^{-2})$	$\frac{1}{3}$	n	2	1	y	n	n	n	n	n	n
$II_{14,2}(3^{+2})$	$\frac{2}{3}$	n	8	3	y	y	y	y	y	n	n
$II_{18,2}(3^{-2})$	$\frac{2}{3}$	n	4	2	y	n	n	n	n	n	n
$II_{26,2}(3^{-2})$	$\frac{3}{3}$	y	9	3	y	y	y	y	y	y	y
$II_{34,2}(3^{-2})$	$\frac{4}{3}$	n	11	6	y	n	n	n	n	n	n
$II_{4,2}(3^{+3})$	$\frac{1}{3}$	n	12	0	y	n	n	n	y	n	n
$II_{8,2}(3^{-3})$	$\frac{1}{3}$	y	6	2	y	y	y	y	y	y	y
	$\frac{2}{3}$	n	18	2	y	y	y	y	y	y	y
$II_{12,2}(3^{+3})$	$\frac{2}{3}$	n	18	6	y	y	y	y	y	n	n
$II_{16,2}(3^{-3})$	$\frac{2}{3}$	n	18	7	y	n	n	n	n	n	n
$II_{20,2}(3^{+3})$	$\frac{3}{3}$	n	27	9	y	y	y	y	y	y	n
$II_{24,2}(3^{-3})$	$\frac{3}{3}$	n	27	10	y	y	y	y	y	n	n
$II_{28,2}(3^{+3})$	$\frac{4}{3}$	n	39	14	y	n	n	n	y	n	n
$II_{32,2}(3^{-3})$	$\frac{4}{3}$	n	33	16	y	y	y	y	y	n	n
$II_{40,2}(3^{-3})$	$\frac{5}{3}$	n	45	21	y	n	n	n	n	n	n
$II_{6,2}(3^{-4})$	$\frac{1}{3}$	n	30	5	y	y	y	y	y	n	n
$II_{10,2}(3^{+4})$	$\frac{1}{3}$	n	24	10	y	n	n	n	n	n	n
	$\frac{2}{3}$	n	48	10	y	n	n	n	y	n	n
$II_{14,2}(3^{-4})$	$\frac{2}{3}$	n	60	20	y	y	y	y	y	n	n
$II_{18,2}(3^{+4})$	$\frac{2}{3}$	n	48	22	y	y	y	y	n	n	n
$II_{22,2}(3^{-4})$	$\frac{3}{3}$	n	81	31	y	n	n	n	y	n	n
$II_{26,2}(3^{+4})$	$\frac{3}{3}$	y	81	34	y	y	y	y	y	y	y
$II_{30,2}(3^{-4})$	$\frac{4}{3}$	n	111	46	y	y	y	y	y	n	n
$II_{34,2}(3^{+4})$	$\frac{4}{3}$	n	105	51	y	n	n	n	n	n	n
$II_{42,2}(3^{+4})$	$\frac{5}{3}$	n	129	63	y	y	y	y	n	n	n

The third column confirms the result that was stated in the introduction. We further observe that in these small cases the arithmetical condition that the equation from the pairing with $E_{k,\overline{\rho}_D}$ has an integral solution is very strong. For bigger discriminant forms this does not seem to be the case, which is not very surprising: if larger m_∞ have to be considered, more different coefficients $a_\gamma(m)$ are included in the equation, which makes integral solutions more probable. In those of the above cases where such a solution exists we usually see that all conditions except for the inequalities seem to be necessary to rule out the existence of F.

We can also rest assured that all lattices in the table comply with the requirements of Theorem 4.3.

Lemma 5.8. *Let L be an even lattice of prime level $p \in \{2, 3\}$ and genus $II_{n,2}(p^{\varepsilon_p n_p})$ with $n > 2$ even. Suppose that L splits a hyperbolic plane (over $\mathbb{Z}$). Then L splits two hyperbolic planes over $\mathbb{Q}$.*

Proof. As mentioned immediately after Theorem 4.3, this is always fulfilled if $n \geq 5$. So we may assume $n = 4$. Then $n_p \leq 4$ by Proposition 5.2. We also have $e\big((n-2)/8\big) = e(1/4) \notin \mathbb{Z}$. Hence the other relations between p, n, ε_p, n_p given in Section 5.2 imply that n_p is odd, $p = 3 \bmod 4$ and $\varepsilon_p\left(\frac{2}{p}\right)(-1)^{(n_p-1)/2} = 1$, i.e. $p = 3$, $n_p \in \{1, 3\}$, $\varepsilon_p = (-1)^{(n_p+1)/2}$.

Let $L \in II_{4,2}(3^{-1})$. Example 1.28 shows that the lattice $A_2 \oplus II_{1,1} \oplus II_{1,1}$ has genus $II_{4,2}(3^{-1})$, too. So by Theorem 1.23, L is isometric to $A_2 \oplus II_{1,1} \oplus II_{1,1}$ and therefore splits two hyperbolic planes already over $\mathbb{Z}$.

Now let $L \in II_{4,2}(3^{+3})$. The rescaled hyperbolic plane $II_{1,1}(3)$ has genus $II_{1,1}(3^{-2})$ as can be shown analogously to Example 1.28 using Example 1.15. As before we get $L = A_2 \oplus II_{1,1}(3) \oplus II_{1,1}$. Now take $II_{1,1}$, $II_{1,1}(3)$ as the set $\mathbb{Z}^2 \subseteq \mathbb{Q}^2$ with the quadratic forms

$$\mathbb{Z}^2 \to \mathbb{Z} : (x, y) \mapsto xy$$

and

$$\mathbb{Z}^2 \to \mathbb{Z} : (x, y) \mapsto 3xy,$$

respectively. Then

$$II_{1,1}(3) \otimes_{\mathbb{Z}} \mathbb{Q} \to II_{1,1} \otimes_{\mathbb{Z}} \mathbb{Q} : (x, y) \mapsto (3x, y)$$

is a bijective isometry. Hence also

$$L \otimes_{\mathbb{Z}} \mathbb{Q} = (A_2 \oplus II_{1,1} \oplus II_{1,1}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

which proves the assertion. □

Finally, let us describe the origin of the automorphic products that have been detected by the algorithm. Most of them are also collected in [Sch14, Theorem 6.27] and all have already been known. For example, by Theorem 2.16 we have $\eta_{1^{-16}2^8} \in \mathcal{M}^!_{-4}(\Gamma_1(2)) = \mathcal{M}^!_{-4}(\Gamma_0(2))$. For $L \in II_{10,2}(2^{+2}_{II})$, $D = L'/L$ the lift $F_{\Gamma_0(2),16\eta_{1^{-16}2^8},\{0\}}$ from $\Gamma(2)$ on 0 has 0-component

$$8\eta_{1^{-16}2^8} = 8 + 128q + 1152q^2 + 7680q^3 + \cdots$$

and nonnegative integral principal part

$$q^{-1/2}e_\gamma$$

where $\gamma \in D$ is determined by $\gamma^2/2 = 1/2 \bmod 1$ [Sch09, Section 8]. Hence $\Psi(F_{\Gamma_0(2),16\eta_{1^{-16}2^8},\{0\}})$ is a holomorphic automorphic product of singular weight on L.

Now let $L \in II_{8,2}(3^{-3})$. Then similarly a holomorphic automorphic product of singular weight on L is given by $\Psi(F_{\Gamma_0(3),9\eta_{1^{-9}3^3},\{0\}})$ [Sch14, Proposition 6.15].

On the other lattices examples can be found as follows.

Lemma 5.9. *Let $D = p^{\varepsilon_p n_p}$ be a discriminant form of prime level p and even p-rank n_p.*

(1) If $\operatorname{sign}(D) = 0 \bmod 8$, there is an isotropic subgroup $H \subseteq D$ such that $|H| = p^{n_p/2}$.

(2) If $p = 2$, there is an isotropic subgroup $H \subseteq D$ such that $|H| = p^{(n_2-2)/2}$.

Proof. First suppose that $\operatorname{sign}(D) = 0 \bmod 8$ and $p = 2$. Then

$$\varepsilon_p = e\big(\operatorname{sign}(D)/8\big)\left(\frac{-1}{p}\right)^{n_p/2} = +1$$

(cf. Section 5.2) and $D = \bigoplus_{j=1}^{n_p/2} 2_{II}^{+2}$, so we may assume $D = 2_{II}^{+2}$. Let $\gamma, \delta \in D$ such that $\gamma\delta = \frac{1}{2} \bmod 1$, $\gamma^2/2 = \delta^2/2 = 0 \bmod 1$. Then $H := \{0, \gamma\}$ is an isotropic subgroup with $|H| = 2 = p^{n_p/2}$.

Now let $p > 2$, $\operatorname{sign}(D) = 0 \bmod 8$. If $n_p = 0 \bmod 4$ or $\left(\frac{-1}{p}\right) = 1$, again we get $\varepsilon_p = +1$ and may assume that $D = p^{+2}$. Then there are generators γ, δ of D such that $\gamma\delta = 0 \bmod 1$, $\gamma^2/2 = \delta^2/2 = a/p \bmod 1$ with $\left(\frac{2a}{p}\right) = +1$. By assumption, $d^2 = -1 \bmod p$ for some $d \in \mathbb{Z}$, so that

$$(d\gamma + \delta)^2/2 = d^2\gamma^2/2 + \delta^2/2 = a(d^2 + 1)/p = 0 \bmod 1,$$

i.e. $d\gamma + \delta$ is isotropic. Now choose $H = \langle d\gamma + \delta \rangle$.

If $n_p = 2 \bmod 4$ and $\left(\frac{-1}{p}\right) = -1$, we get $\varepsilon_p = -1$ and $D = \bigoplus_{j=1}^{n_p/2} p^{-2}$. Hence now we may assume that $D = p^{-2}$. Let γ, δ be generators of D such that $\gamma\delta = 0 \bmod 1$, $\gamma^2/2 = a/p \bmod 1$, $\delta^2/2 = b/p \bmod 1$ where $\left(\frac{2a}{p}\right) = +1$, $\left(\frac{2b}{p}\right) = -1$. Then $c^2 = 2a \bmod p$ and $d^2 = -2b \bmod p$ for some $c, d \in \mathbb{Z}$. Hence

$$(d\gamma + c\delta)^2/2 = (ad^2 + bc^2)/p = (-2ab + 2ba)/p = 0 \bmod 1,$$

so $d\gamma + c\delta$ is isotropic. Now choosing $H = \langle d\gamma + c\delta \rangle$ finishes the proof of (1).

For (2) just write $D = 2_{II}^{\pm 2} \oplus 2_{II}^{+(n_2-2)}$ and apply (1) to the second summand. □

Corollary 5.10. *Let L be an even lattice of prime level p and genus $II_{n,2}(p^{\varepsilon_p n_p})$ with $n > 2$ even.*

(1) If n_p is even and $n = 26$, L carries a holomorphic automorphic product of singular weight.

(2) *If* $p = 2$ *and* $n = 10$, L *carries a holomorphic automorphic product of singular weight.*

Proof. As for (1), by Lemma 5.9(1) there is an isotropic subgroup $H \subseteq D$ such that $|H| = \sqrt{|D|}$. Recall from Example 4.6 that $\Psi(1/\Delta)$ is a holomorphic automorphic product of singular weight on $II_{26,2}$. In particular, $1/\Delta$ has nonnegative principal part and constant coefficient 24. These properties are preserved when applying Theorem 3.14 with $D_H = H^\perp/H \in II_{26,2}$, $F_{D_H} = 1/\Delta$ to induce an element $F \in \mathcal{M}^!_{1-n/2,\rho_D}$.

For $p = 2$, $n = 10$ by Lemma 5.9(2) there is an isotropic subgroup $H \subseteq D$ with $|H| = \frac{1}{2}\sqrt{|D|}$. Now as before the assertion follows from Theorem 3.14 with $D_H = 2_{II}^{+2}$, $F_{D_H} = F_{\Gamma_0(2),16\eta_1{-16}_2 8,\{0\}}$. □

Conclusive remarks

A few comments on the significance of the presented method and on possible future improvements shall conclude this work.

One of its purposes was to prove that there are no previously unknown holomorphic automorphic products of singular weight within the given bounds and to provide additional information on these cases. This aim was no doubt achieved, although it is still to be clarified whether the collected data is fit to give rise to new conjectures concerning general laws.

At this point, two weak spots of the program in use become visible. In practice its current version is rather demanding with regard to both computing power and memory. The cardinality of the discriminant form and the upper bound on the signature grow exponentially and linearly with the p-rank, respectively. The number of lifts from $\Gamma_1(p)$ to be computed, the most time-consuming part of the algorithm, in turn increases linearly in both of these. This is one of the reasons why the available time and hardware did not allow to list results for discriminant forms with significantly more than 100 elements. The other weakness is that, with the given software, it was not possible to run the complete process within one environment. Data had to be passed from the main program on to the solver, which does not only hamper further automation, but also restricts precision since the file format stores the problem data as floating point numbers. So if some constraint cannot be formulated with small integer coefficients, rounding errors are introduced at this point.

One desirable future improvement would therefore consist in a good deal of code optimization. Probably an experienced programmer could refine the rather naive Sage implementation written for this thesis with a bit of effort. Next, the system of equations obtained from obstruction theory is vastly overdetermined, so in particular here some symmetries of the problem might hopefully be exploited. Then rewriting everything in a more efficient programming language and in a single program, thereby perhaps even adapting the solving technique in the last step to this specific type of problem, and afterwards running it on a more powerful machine should allow to go much further.

The other main purpose of the thesis was to provide the first purely algorithmic method to determine whether or not any lattice with given discriminant form and prime level carries a holomorphic automorphic product of singular weight. More precisely, so far it was expected that, thanks to the bounds on signature and pole orders, one could do so by working out the obstruction spaces [Sch14]. However, only now we have a detailed description of how this can be carried out systematically, which might serve as a pattern for future generalizations.

And indeed generalizations are to be done also on the theoretical side. Throughout the process we had to assume a lot of premises. Currently, apart from software issues, the bottleneck is the construction of bases of spaces of scalar valued cusp forms. Doing the same for some other primes should be possible, though already here higher dimensions and increasing numbers of cusps will make the work tedious. So with respect to this point, the ultimate goal could be an algorithm to construct these bases for general level. The hyperbolic plane condition is rooted quite deep in the theory, but fortunately it is not such a hard restriction either. Finally, a great step would be to get rid of the prime level condition. This seems difficult, but certainly work on a generalization, at least to squarefree level or prime powers, is already in progress.

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und alle benutzten Quellen und sonstigen Hilfsmittel angegeben habe. Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

Ort, Datum

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