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# ON SPECIAL QUADRATIC BIRATIONAL TRANSFORMATIONS OF A PROJECTIVE SPACE

TESI DI DOTTORATO DI RICERCA IN MATEMATICA
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PH.D. THESIS IN MATHEMATICS (XXIII CYCLE)

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ABSTRACT

A birational map from a projective space onto a not too much singular projective variety with a single irreducible non-singular base locus scheme (*special birational transformation*) is a rare enough phenomenon to allow meaningful and concise classification results.

We shall concentrate on transformations defined by quadratic equations onto some varieties (especially projective hypersurfaces of small degree), where quite surprisingly the base loci are interesting projective manifolds appearing in other contexts; for example, exceptions for adjunction theory, small degree or small codimensional manifolds, Severi or more generally homogeneous varieties.

In particular, we shall classify:

- quadro-quadric transformations into a quadric hypersurface;
- quadro-cubic transformations into a del Pezzo variety;
- transformations whose base locus (scheme) has dimension at most three.

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# Contents

# Introduction

Consider, on a complex projective space $\mathbb{P}^n$, a fixed component free sublinear system $\sigma \subset |\mathscr{O}_{\mathbb{P}^n}(d_0)|$, of dimension $N \geq n$, such that the associated rational map $\varphi = \varphi_\sigma : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ is birational onto its image and moreover such that the image is not too much singular. Understanding all such linear systems (or the corresponding birational transformations) is clearly a too ambitious goal. Already for $N = n$ the Cremona group of all these transformations is a very complicated object. From now on, assume $d_0 = 2$ and denote by $d$ the degree of the linear system giving the inverse to $\varphi$. We then say that the transformation $\varphi$ is of type $(2,d)$. Moreover, assume that $\varphi$ is *special*, i.e. its base locus scheme (also called center) is smooth and connected. The first interesting case is when $N = n$, i.e. that of special quadratic Cremona transformations.

## Special Cremona transformations

The first general results were obtained by B. Crauder and S. Katz in [CK89] (see also [CK91] and [Kat87]), by classifying all special Cremona transformations whose base locus has dimension at most two. In particular, they obtained that the base locus of a quadratic transformation of this kind (which is nondegenerate in $\mathbb{P}^n$) is one of the following:

✓ a quintic elliptic curve in $\mathbb{P}^4$;

✓ the Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

✓ a septic elliptic scroll in lines embedded in $\mathbb{P}^6$;

✓ the plane blown-up at eight points and embedded in $\mathbb{P}^6$ as an octic surface.

The second general result was obtained by L. Ein and N. Shepherd-Barron in [ESB89]: the base locus of a special Cremona transformation of type $(2,2)$ is a Severi variety. Moreover F. L. Zak (see [LV84]) had shown that there are just four Severi varieties:

- the Veronese surface $\nu_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;
- the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$;
- the Plücker embedding of $\mathbb{G}(1,5)$ in $\mathbb{P}^{14}$;
- the 16-dimensional Cartan variety $E_6$ in $\mathbb{P}^{26}$.

The next step was taken by F. Russo in [Rus09], who observed that base loci of special Cremona transformations of type $(2,d)$ are of a very peculiar type, the so called *quadratic entry locus* varieties. He classified some cases (for example with $n$ odd, or of type $(2,3)$) and suggested that such base loci are subject to very strong restrictions.

On special quadratic Cremona transformations whose base locus has dimension three, K. Hulek, S. Katz and F. O. Schreyer in [HKS92] provided an example, the only example so far known. Later, M. Mella and F. Russo in the unpublished paper [MR05], collected a series of ideas and remarks on the study of these transformations. One of these ideas was to apply the Castelnuovo theory to the general zero-dimensional linear section of the base locus. This idea is central to the present thesis.

## The next case

The first main goal of the thesis is to deal with the "next case", $N = n+1$, under the assumption that the image of the special transformation of type $(2,d)$ is a sufficiently regular hypersurface. Under such hypotheses, we are still able to show that the base locus is a quadratic entry locus variety; moreover, we compute the dimension and the secant defect of the base locus in terms of the other numerical invariants: $n$, $d$ and the degree of the hypersurface image.

The first and easiest example of such a transformation is the inverse of a stereographic projection of a quadric. This example can be characterized in various ways. For example, it is the only case in which the base locus is degenerate.

In this direction, the main results of the thesis are three:

- ✠ Complete classification when the type is $(2,2)$ and the image is a quadric: the base locus is a hyperplane section of a Severi variety.

- ✠ Complete classification when the type is $(2,3)$ and the image is a cubic: the base locus is a three-dimensional quadric blown-up at five points and embedded in $\mathbb{P}^8$.

- ✠ (Almost) complete classification when the dimension of the base locus is at most three.

On the first item, the existence of examples is clear and the difficulty is to prove that they are the only ones. This is done in several steps, in order to prove that the inverse transformation is still special. In the third item, we wrote "(Almost) complete classification" because in one case we do not know if it really exists. However, we are able to say that the case exists if one proves that a linearly normal scroll in lines over the Hirzebruch surface $\mathbb{F}_1$, embedded in $\mathbb{P}^8$ as a variety of degree 11 and sectional genus 5 (whose existence has been established by A. Alzati and G. M. Besana in [AB10]) is also cut out by quadrics. On the second item, by using some results of P. Ionescu and F. Russo in [Rus09] and [IR10], we deduce that the base locus is three-dimensional and so we apply the previously obtained classification. Next, we also compute the possible numerical invariants for transformations of type $(2,2)$ into a cubic and a quartic hypersurface.

## Towards the general case

The second main goal of the thesis is to deal with a more general case, that is when there are no restrictions on $N$, but however the image $\overline{\varphi(\mathbb{P}^n)} \subseteq \mathbb{P}^N$ of the transformation is a sufficiently regular variety. We study these transformations either keeping the dimension of the base locus small or fixing $d$ and another numerical invariant, the *coindex* of the image.

Recall that in [Sem31], J. G. Semple constructed transformations $\mathbb{P}^{2m-2} \dashrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ of type $(2,2)$ (resp. type $(2,1)$) having as image the Grassmannian $\mathbb{G}(1,m) \subset \mathbb{P}^{\binom{m+1}{2}-1}$ and having as base locus a nondegenerate (resp. degenerate) rational normal scroll. Further, F. Russo and A. Simis in [RS01], have characterized these examples as the only special birational transformations of type $(2,2)$ (resp. type $(2,1)$) into the Grassmannian of lines in projective space.

Note that in the Semple's examples, the image of the transformation is smooth. However, the smoothness of the image is a very rare phenomenon and therefore, in order not to exclude relevant cases, we only require that the image is "sufficiently regular". On the other hand, this assumption on the image is reasonable to restrict the classification in a confined meaningful list.

By applying techniques and results obtained for the case $N = n+1$, we extend some of our results. More precisely we obtain:

✠ Classification when the type is $(2,3)$ and the image is a "del Pezzo variety": the image is a cubic hypersurface or the base locus is either a scroll in lines over a quadric surface or a quadric surface fibration over a line.

✠ Classification of all transformations when the dimension of the base locus is at most three: there are (at most) 33 types of such transformations.

Here, we determine all possible cases mainly by applying the M. Mella and F. Russo's idea aforementioned and the classification of smooth varieties of low degree, but the main difficulty is to exhibit examples. Even in the simplest cases, the number of calculations is so huge that the use of a computer algebra system is indispensable. In some cases we are able to say that there is a transformation just as we wish, except for the fact that we do not know if the image satisfies all our assumptions.

We point out that, as a consequence, one can deduce that there are (at most) two types of special quadratic Cremona transformations having three-dimensional base locus: we have $n = N = 8$ and the base locus is either

✓ the projection from a point of a Fano variety (here we have the K. Hulek, S. Katz and F. O. Schreyer's example) or

✓ a scroll in lines over a surface (here we do not know examples).

From F. Russo's results in [Rus09] it follows that these transformations are the only special Cremona transformations of type $(2,5)$.

In an appendix, we show how the techniques used may be adapted to the case in which the dimension of the base locus is greater than three, focusing to the case of dimension four. We also classify these transformations, but with $N$ large. It does appear clear that the complexity of the objects increases with the decreasing of $N-n$.

## Two-sides principle

From the thesis it follows a two-sides principle: on one hand, base loci of such special birational transformations are very particular (they are manifolds satisfying some strong numerical and geometric conditions) and further, at least where we were able to construct them, their properties may be described quite precisely. Conversely, all these special manifolds seem to appear as base loci of conveniently chosen transformations.

## Organization of the thesis

In the first chapter we recall basic facts on geometric objects as the secant variety and the tangential projections of a projective variety.

The second chapter is dedicated to presenting some well-known classes of projective varieties. In particular, we treat the class of quadratic entry locus varieties, largely studied by P. Ionescu and F. Russo.

In the third chapter, we outline some well-known extensions of Castelnuovo's results. These are applied to study transformations having base locus of small dimension.

The fourth chapter is an introduction to the subject of the thesis. We reinterpret the well-known example of the stereographic projection and treat the case of transformations of $\mathbb{P}^n$ into a quadric, with $n \leq 4$ and having reduced base loci.

The fifth and sixth chapter are the main part of the thesis. There we prove the main results on special quadratic birational transformations aforementioned. Much of these two chapters is also contained in the papers [Sta12a] and [Sta12b].

# Chapter 1

# Basic tools

For much of the material in this chapter, the main references are [Zak93] and [Rus10]. Unless otherwise specified, we shall keep the following:

**Assumption 1.1.** $X \subset \mathbb{P}^n$ is an irreducible nondegenerate (i.e. not contained in a hyperplane) $r$-dimensional projective complex algebraic variety.

## 1.1 Secant and tangent loci

### 1.1.1 Secant variety

Let $X \subset \mathbb{P}^n$ be as in Assumption 1.1 and let $Y \subset \mathbb{P}^n$ be an irreducible projective variety. We put

$$\Delta_Y = \{(y,x) \in Y \times X : y = x\},$$

$$S^0_{Y,X} = \{(y,x,z) \in (Y \times X \setminus \Delta_Y) \times \mathbb{P}^n : z \in \langle y,x \rangle\},$$

and denote by $S_{Y,X}$ the closure of $S^0_{Y,X}$ in $Y \times X \times \mathbb{P}^n$. So we obtain the diagram:

$$\begin{tikzcd} & & S_{Y,X} \arrow[dl, "\pi_{Y,X}"'] \arrow[dr, "\pi_{\mathbb{P}^n}"] & \\ & Y \times X \arrow[dl, "p_Y"'] \arrow[dr, "p_X"] & & \mathbb{P}^n \\ Y & & X & \end{tikzcd} \tag{1.1}$$

**Definition 1.2.** The *join* of $Y$ and $X$, $S(Y,X)$, is defined as the scheme-theoretic image of $\pi_{\mathbb{P}^n}$, i.e.

$$S(Y,X) = \pi_{\mathbb{P}^n}(S_{Y,X}) = \overline{\bigcup_{(y,x)\in Y\times X\setminus\Delta_Y} \langle y,x \rangle}.$$

Note that $S_{Y,X}$ is an irreducible variety of dimension $\dim(Y)+\dim(X)+1$, hence $S(Y,X)$ is also irreducible of dimension

$$\dim(S(Y,X)) \leq \dim(Y)+\dim(X)+1. \tag{1.2}$$

An essential tool for studying joins of varieties is the so-called Terracini Lemma (see also [Rus10, Theorem 1.3.1]):

**Theorem 1.3** (Terracini Lemma)**.** *Let $Y,X \subset \mathbb{P}^n$ be as above. There exists a nonempty open subset $U$ of $S(Y,X)$ such that for any $p \in U$, $(y,x) \in Y \times X \setminus \Delta_Y$, $p \in \langle y,x\rangle$, we have*

$$T_p(S(Y,X)) = \langle T_y(Y), T_x(X)\rangle.$$

**Definition 1.4.** In the case when $Y = X$, we put $\mathrm{Sec}(X) := S(X,X)$ and call it *secant variety* of $X$. A line $l$ is said to be a *secant* of $X$ if the length of the scheme $X \cap l$ is at least 2; in these terms, $\mathrm{Sec}(X)$ is the union of all secant lines of $X$. The *secant defect* of $X$ is defined as the nonnegative integer

$$\delta(X) = 2\dim(X)+1-\dim(\mathrm{Sec}(X)) \tag{1.3}$$

and $X$ is called *secant defective* if $\delta(X) > 0$.

*Remark* 1.5. The definition of secant variety can be given in a more general context. Indeed for any locally closed subscheme $B \subset \mathbb{P}^n$ we can consider the Hilbert scheme $\mathrm{Hilb}_2(B)$ of its 0-dimensional subschemes of length 2. For each $Z \in \mathrm{Hilb}_2(B)$ the evaluation map $H^0(B,\mathscr{O}_B(1)) \to H^0(Z,\mathscr{O}_Z(1))$ is obviously surjective, therefore its kernel defines a secant line $\langle Z\rangle \subset \mathbb{P}^n$. This defines a $\mathbb{P}^1$-bundle over $\mathrm{Hilb}_2(B)$

$$\mathscr{S}(B) := \{(Z,p) \in \mathrm{Hilb}_2(B) \times \mathbb{P}^n : p \in \langle Z\rangle\} \longrightarrow \mathrm{Hilb}_2(B)$$

and the *secant scheme* of $B$, $\mathrm{Sec}(B)$, is defined as the scheme-theoretic image of the projection $\mathscr{S}(B) \to \mathbb{P}^n$.

### 1.1.2 Contact loci

Consider the diagram (1.1) with $Y = X$, defining the secant variety $\mathrm{Sec}(X)$, and let $p \in \mathrm{Sec}(X) \setminus X$ be a point. Put

$$\begin{array}{rcl} L_p(X) & := & \pi_{\mathbb{P}^n}(\pi_{X,X}^{-1}(\pi_{X,X}(\pi_{\mathbb{P}^n}^{-1}(p)))) \\ & = & \bigcup\{l : l \text{ secant line of } X \text{ through } p\}, \end{array}$$

$$\begin{array}{rcl} \Sigma_p(X) & := & \pi_X(\pi_{\mathbb{P}^n}^{-1}(p)) = L_p(X) \cap X \\ & = & \overline{\{x \in X : \exists x' \in X \text{ with } x \neq x' \text{ and } p \in \langle x,x'\rangle\}}, \end{array}$$

where $\pi_X = p_X \circ \pi_{X,X}$.

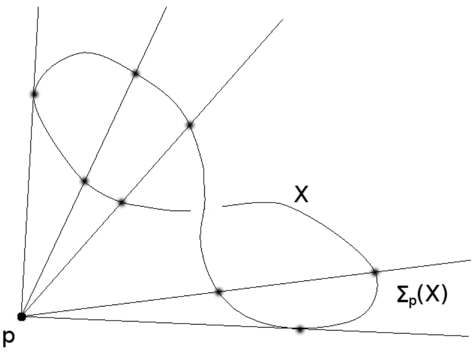


Figure 1.1: Entry locus of $X$.

**Definition 1.6.** $\Sigma_p(X)$ is called the *entry locus* of $X$ with respect to $p$.

Note that $L_p(X)$ is a cone over $\Sigma_p(X)$ of vertex $p$, hence $\dim(L_p(X)) = \dim(\Sigma_p(X)) + 1$. If $p \in \mathrm{Sec}(X) \setminus X$ is a general point, then $\Sigma_p(X)$ is reduced and equidimensional of dimension $\delta(X)$.

**Definition 1.7.** The *tangential contact locus* of $X$ with respect to the general point $p$ is defined as

$$\Gamma_p(X) := \overline{\{x \in \mathrm{reg}(X) : T_x(X) \subseteq T_p(\mathrm{Sec}(X))\}}.$$

By Terracini Lemma it follows that $\Sigma_p(X) \subseteq \Gamma_p(X)$ and it is possible to prove that for $x, x' \in X$ general points and $p \in \langle x, x' \rangle$ general point, the irreducible components of $\Gamma_p(X)$ through $x$ and $x'$ are uniquely determined and have the same dimension, independent of $p$. Denoting this dimension by $\gamma = \gamma(X)$, we have $\delta(X) \leq \gamma(X)$. See also [Rus10, Definition 2.3.3] and [CC10, Definition 3.4].

### 1.1.3 $J$-tangency

Let $Y \subseteq X$ be a subvariety ($X$ as in Assumption 1.1) and consider the diagram (1.1).

**Definition 1.8.** For a point $y \in Y$, the *(relative) tangent star* to $X$ with respect to $Y$ at $y$, is defined as

$$T_y^*(Y,X) := \pi_{\mathbb{P}^n}(\pi_{Y,X}^{-1}((y,y))),$$

and put

$$T^*(Y,X) := \bigcup_{y \in Y} T_y^*(Y,X).$$

We have $T_y^*(Y,X) \subseteq T_y(X)$ and, if $y \in Y \setminus \mathrm{sing}(X)$, then $T_y^*(Y,X) = T_y(X)$; in particular, if $Y \subseteq X \setminus \mathrm{sing}(X)$, then

$$T^*(Y,X) = T(Y,X) := \bigcup_{y \in Y} T_y(X).$$

**Theorem 1.9** ([Zak93]). *It holds one of the following two conditions:*

1. $\dim(T^*(Y,X)) = \dim(S(Y,X)) - 1 = \dim(Y) + \dim(X)$;
2. $T^*(Y,X) = S(Y,X)$.

**Definition 1.10.** If $L \subset \mathbb{P}^n$ is a linear space, we say that $L$ is tangent to $X$ along $Y$ if $L \supseteq T(Y,X)$; we say that $L$ is $J$-tangent to $X$ along $Y$ if $L \supseteq T^*(Y,X)$.

**Corollary 1.11** (Zak's Theorem on tangencies, [Zak93]). *Notation as above. If $L \subset \mathbb{P}^n$ is a linear space $J$-tangent to $X$ along $Y$, then*

$$\dim(Y) + \dim(X) \leq \dim(L).$$

*Proof.* See also [Zak93, I Corollary 1.8] and [Rus10, Theorem 2.2.3]. Without loss of generality we can suppose that $Y$ is irreducible. By hypothesis $T^*(Y,X) \subseteq L$, but $S(Y,X) \nsubseteq L$ (because $X \subseteq S(Y,X)$ and $X$ is nondegenerate). Thus, by Theorem 1.9, we have $\dim(Y) + \dim(X) = \dim(T^*(Y,X)) \leq \dim(L)$. □

### 1.1.4 Gauss map

Recall that the Gauss map of $X$ is the rational map $\mathscr{G}_X : X \dashrightarrow \mathbb{G}(r,n)$, which sends the point $x \in \operatorname{reg}(X)$ to the tangent space $T_x(X) \in \mathbb{G}(r,n)$. For a general point $x \in \operatorname{reg}(X)$, the closure of the fiber of $\mathscr{G}_X$ at $\mathscr{G}_X(x)$ is

$$\overline{\mathscr{G}_X^{-1}(\mathscr{G}_X(x))} = \overline{\{y \in \operatorname{reg}(X) : T_y(X) = T_x(X)\}}.$$

**Theorem 1.12.** *For $x \in X$ general point, $\overline{\mathscr{G}_X^{-1}(\mathscr{G}_X(x))}$ is a linear space.*

**Corollary 1.13.** *If $X$ is smooth, then $\mathscr{G}_X : X \to \mathscr{G}_X(X)$ is a birational morphism.*

For details and proof of Theorem 1.12, we refer the reader to [Zak93, I §2]. Corollary 1.13 is instead a straightforward application of Corollary 1.11 and Theorem 1.12.

## 1.2 Tangential projections

For a general point $x \in X$ consider the projection of $X$ from the tangent space $T_x(X) \simeq \mathbb{P}^r$ onto a linear space $\mathbb{P}^{n-r-1}$ skew to $T_x(X)$,

$$\tau_{x,X} : X \subset \mathbb{P}^n \dashrightarrow \overline{\tau_{x,X}(X)} = W_{x,X} \subseteq \mathbb{P}^{n-r-1}.$$

From Terracini Lemma it follows (see also [Rus10, Proposition 1.3.8]):

**Proposition 1.14.** *$W_{x,X} \subseteq \mathbb{P}^{n-r-1}$ is an irreducible nondegenerate variety of dimension $r - \delta(X)$.*

In §1.2.1 we introduce a useful concept for studying tangential projections of secant defective varieties having secant variety that doesn't fill up the whole ambient space.

### 1.2.1 Second fundamental form

Consider the blow-up of $X$ at the general point $x$, $\pi_x : \mathrm{Bl}_x(X) \to X$, and denote by $E = E_X = E_{x,X} = \mathbb{P}^{r-1}$ its exceptional divisor and by $H$ a divisor of the linear system $|\pi_x^*(\mathscr{O}_X(1))|$. Since $X$ is not linear, it is defined the rational map

$$\phi : E \dashrightarrow W_{x,X} \subseteq \mathbb{P}^{n-r-1}.$$

**Definition 1.15.** The linear system associated to $\phi$, i.e. $|H-2E|_{|E} \subseteq |-2E|_{|E} = |\mathscr{O}_{\mathbb{P}^{r-1}}(2)|$, is called *second fundamental form* of $X$ (at the point $x$) and it is denoted with $|II_{x,X}|$.

Of course, $\dim(|II_{x,X}|) \leq n-r-1$ and the image of $\phi = \phi_{|II_{x,X}|}$ is contained in $W_{x,X}$.

**Proposition 1.16.** *If $X$ is smooth, secant defective and* $\mathrm{Sec}(X) \subsetneq \mathbb{P}^n$*, then* $\overline{\phi_{|II_{x,X}|}(E)} = W_{x,X}$ *and in particular* $\dim(|II_{x,X}|) = n-r-1$.

See [Rus10, Proposition 2.3.2] for a proof of Proposition 1.16. In the following we shall denote by $B_{x,X} = \mathrm{Bs}(|II_{x,X}|)$ the base locus of the second fundamental form.

### 1.2.2 Hilbert scheme of lines passing through a point

Let $X \subseteq \mathbb{P}^n$ be a closed subscheme, $x \in X$.

**Definition 1.17.** The *Hilbert scheme of lines contained in $X$ and passing through $x$* is a scheme (over $\mathbb{C}$ as always), $\mathscr{L}_{x,X}$, together with a closed subscheme $\chi_{x,X} \hookrightarrow X \times \mathscr{L}_{x,X}$ (called *universal family*) such that the fibers of the projection $\pi_{x,X} : \chi_{x,X} \to \mathscr{L}_{x,X}$ (which are naturally immersed in $X \subseteq \mathbb{P}^n$) are lines passing through $x$. Moreover $(\mathscr{L}_{x,X}, \chi_{x,X})$ must satisfy the following universal property: for each scheme $\mathscr{Y}$ and for each closed subscheme $Z \hookrightarrow X \times \mathscr{Y}$, having the property that the fibers of the projection $Z \to \mathscr{Y}$ are lines passing through $x$, there exists a unique morphism $f = f_{(\mathscr{Y},Z)} : \mathscr{Y} \to \mathscr{L}_{x,X}$ such that we have the diagram of fibred products:

$$\begin{array}{ccccc} Z & \hookrightarrow & X \times \mathscr{Y} & \longrightarrow & \mathscr{Y} \\ \downarrow & & \downarrow & & \downarrow f \\ \chi_{x,X} & \hookrightarrow & X \times \mathscr{L}_{x,X} & \longrightarrow & \mathscr{L}_{x,X} \end{array} \tag{1.4}$$

As a consequence of the general result in [Kol96, I Theorem 1.4] we obtain that $\mathscr{L}_{x,X}$ exists and it is a projective scheme. Moreover, if $x \in \mathrm{reg}(X)$, there exists a natural closed embedding of $\mathscr{L}_{x,X}$ into the exceptional divisor $E_{x,X}$ (see [Rus12]).

If $l$ is a line contained in $X$ and passing through $x$, we denote by $[l]$ the corresponding element of $\mathscr{L}_{x,X}$ (i.e. $l = \pi_{x,X}^{-1}([l])$). Applying [Deb01, Proposition 4.14] and [Kol96, I Definition 2.6, Theorem 2.8] and arguing as in [Rus12, Proposition 1.1(1)], one can show the following:

**Proposition 1.18.** *Let $X \subseteq \mathbb{P}^n$ be a generically reduced closed subscheme, $x \in X$ a general point. Then*

$$\mathrm{sing}(\mathscr{L}_{x,X}) \subseteq S_{x,X} := \{[l] \in \mathscr{L}_{x,X} : l \cap \mathrm{sing}(X) \neq \emptyset\}.$$

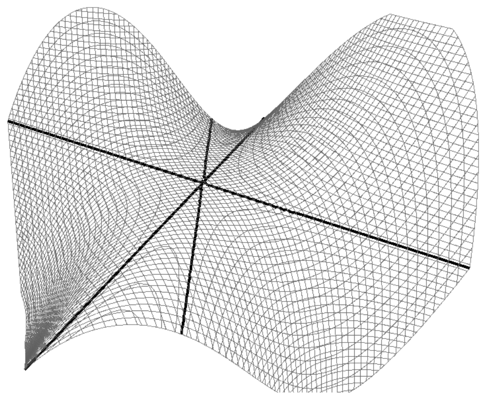

Figure 1.2: Hilbert scheme of lines through a point.

In the following, let $X$ be as in Assumption 1.1. Note that, in general $\mathscr{L}_{x,X}$ is neither irreducible nor reduced. However, from Proposition 1.18, it follows that if $X$ is smooth and $x \in X$ is a general point, then $\mathscr{L}_{x,X}$ is smooth; moreover, for $[l] \in \mathscr{L}_{x,X}$, we have

$$\dim_{[l]}(\mathscr{L}_{x,X}) = H^0(\mathbb{P}^1, \mathscr{N}_{l,X}(-1)) = H^0(\mathbb{P}^1, \mathscr{T}_X|_l(-1)) - 2 = -K_X \cdot l - 2. \tag{1.5}$$

The relation between $\mathscr{L}_{x,X}$ and $B_{x,X}$ is given by the following:

**Proposition 1.19.** *Let $X \subset \mathbb{P}^n$ be as in Assumption 1.1 and let $x \in \operatorname{reg}(X)$. Then there exists a natural closed embedding of $\mathscr{L}_{x,X}$ into $B_{x,X}$, which is an isomorphism if $X$ is defined by quadratic forms.*

For the proof of Proposition 1.19, we refer the reader to [Rus12, Corollary 1.6].

# Chapter 2

# Some special varieties

In this chapter we introduce some well-known classes of varieties which play an important role in the rest of the thesis. Unless otherwise specified, we shall keep Assumption 1.1.

## 2.1 Fano varieties

**Definition 2.1.** The variety $X$ is called a *Fano variety* if it is smooth and its anticanonical bundle $-K_X$ is ample; in such case the *index* of $X$ is

$$i(X) := \sup\{m \in \mathbb{Z} : -K_X \sim mL \text{ for some ample divisor } L\},$$

while the *coindex* of $X$ is

$$c(X) := \dim(X) + 1 - i(X).$$

*Remark* 2.2. If $X$ is a Fano variety, since $X$ is smooth, the group $\mathrm{Pic}(X)$ is torsion free and therefore there exists a unique (ample) divisor $L$ such that $-K_X \sim i(X)L$.

We are interested in the case in which $-K_X \sim i(X)H_X$, where $H_X$ is a hyperplane section of $X \subset \mathbb{P}^n$. We also say that a Fano variety $X \subset \mathbb{P}^n$ is *of the first species* if $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle \mathscr{O}_X(1)\rangle$.

*Remark* 2.3. If $X \subset \mathbb{P}^n$ is a Fano variety, from Kodaira Vanishing Theorem and Serre Duality (see e.g. [Har77, III Corollary 7.7, Remark 7.15], [Fuj90, Theorem 0.4.11, Corollary 0.4.13]) it follows $H^i(X, \mathscr{O}_X) = 0$ for $i \geq 1$. Hence from the cohomology sequence

$$\cdots \to H^1(X, \mathscr{O}_X) \to \underbrace{H^1(X, \mathscr{O}_X^*)}_{\simeq \mathrm{Pic}(X)} \to H^2(X, \mathbb{Z}) \to H^2(X, \mathscr{O}_X) \to \cdots$$

of the exponential sheaf sequence (see [GH78, p. 37] or [Har77, p. 446]) we see that the rank of the Picard group $\mathrm{Pic}(X)$ equals the second Betti number $b_2(X)$. Moreover note that, from Barth-Larsen Theorem [Lar73], each Fano variety is of the first species if $2r - n \geq 2$.

**Example 2.4.** Simple examples of Fano varieties are: projective spaces; smooth complete intersections in $\mathbb{P}^n$ defined by equations of degrees $d_1, \ldots, d_s$ with $d_1 + \cdots + d_s \leq n$; finite products of Fano varieties.

**Theorem 2.5** (Kobayashi-Ochiai [KO73]). *If $X$ is a Fano variety, then $c(X) \geq 0$ and*

1. *if $c(X) = 0$, then $X$ is a projective space;*
2. *if $c(X) = 1$, then $X$ is a quadric hypersurface.*

Below we treat Fano varieties of coindex 2 and 3.

### 2.1.1 Del Pezzo varieties

Let $X \subset \mathbb{P}^n$ be as in Assumption 1.1 and further suppose that it is smooth and linearly normal. Denote by $r$, $\lambda$, $g$ and $\Delta_X$, respectively, the dimension, the degree, the sectional genus and the $\Delta$-genus of $X$ (the latter defined as $\Delta_X = r + \lambda - n - 1$).

**Proposition 2.6** ([Fuj90]). *The following are equivalent:*

- $\Delta_X = g = 1$*;*
- $K_X \in |\mathscr{O}_X(1-r)|$.

If one of the equivalent conditions of Proposition 2.6 holds, then $X$ is called a *del Pezzo variety*. A del Pezzo surface, different from $\mathbb{P}^1 \times \mathbb{P}^1$, can be obtained from $\mathbb{P}^2$ by blowing up several points successively; these points are not infinitely near and no three of them are collinear and no six of them lie on a conic (see [Fuj90, I §8]). Del Pezzo varieties of higher dimension are classified by the following:

**Theorem 2.7** ([Fuj90]). *Let $X \subset \mathbb{P}^n$ be a del Pezzo variety as above, with $r \geq 3$. Then $X$ is projectively equivalent to one of the following:*

($\lambda = 3$) *a cubic hypersurface;*

($\lambda = 4$) *a complete intersection of two quadric hypersurfaces;*

($\lambda = 5$) *a linear section of the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$;*

($\lambda = 6$) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *or* $\mathbb{P}^2 \times \mathbb{P}^2$ *or* $\mathbb{P}(\mathscr{T}_{\mathbb{P}^2})$ *for the tangent bundle* $\mathscr{T}_{\mathbb{P}^2}$ *of* $\mathbb{P}^2$*;*

($\lambda = 7$) *the blowing-up of $\mathbb{P}^3$ at a point;*

($\lambda = 8$) *the Veronese immersion $\nu_2(\mathbb{P}^3) \subset \mathbb{P}^9$.*

### 2.1.2 Mukai varieties

Let $X \subset \mathbb{P}^n$ be as in Assumption 1.1 and further suppose that it is smooth and linearly normal. As above, denote by $r$, $\lambda$ and $g$, respectively, dimension, degree and sectional genus of $X$. $X$ is called a *Mukai variety* if one of the following equivalent conditions holds:

- $X$ has a smooth curve section $C$ embedded by the canonical linear system $|K_C|$;
- $K_X \in |\mathscr{O}_X(2-r)|$.

Note that for a Mukai variety $X$, by sectional genus formula we get the relation:

$$\lambda = 2\,(g-1)\,. \tag{2.1}$$

**Theorem 2.8** ([Muk89]). *The following varieties (denoted by $X$ and embedded in $\langle X\rangle \simeq \mathbb{P}^n$) are examples of Mukai varieties of the first species (where $g,r,n$ are indicated by side):*

($g=6$, $r=6$, $n=10$) *A smooth intersection $X$ of a quadric hypersurface in $\mathbb{P}^{10}$ with the cone over $\mathbb{G}(1,4)\subset\mathbb{P}^9$.*

($g=7$, $r=10$, $n=15$) *The spinorial variety $X=S^{10}\subset\mathbb{P}^{15}$.*

($g=8$, $r=8$, $n=14$) *The Grassmannian $X=\mathbb{G}(1,5)\subset\mathbb{P}^{14}$.*

($g=9$, $r=6$, $n=13$) *Let $U$ be a 6-dimensional vector space (over $\mathbb{C}$) and $F$ be a nondegenerate skew-symmetric bilinear form on $U$. All 3-dimensional subspaces $W$ of $U$ with $F(W,W)=0$ form a smooth variety $X\subset\mathbb{G}(2,5)\subset\mathbb{P}^{19}$.*

($g=10$, $r=5$, $n=13$) *Let $U$ be a 7-dimensional vector space and $F$ be a nondegenerate skew-symmetric 4-linear form on $U$. All 5-dimensional subspaces $W$ of $U$ with $F(W,W,W,W)=0$ form a smooth variety $X\subset\mathbb{G}(4,6)\subset\mathbb{P}^{20}$.*

($g=12$, $r=3$, $n=13$) *Let $U$ be a 7-dimensional vector space and $F_1,F_2,F_3$ be three linearly independent skew-symmetric bilinear forms on $U$. All 3-dimensional subspaces $W$ of $U$ with $F_1(W,W)=F_2(W,W)=F_3(W,W)=0$ form a subvariety $X$ of $\mathbb{G}(2,6)\simeq G(3,U)\subset\mathbb{P}^*(\bigwedge^3 U)\simeq\mathbb{P}^{34}$; if the subspace $F_1\wedge U^\vee+F_2\wedge U^\vee+F_3\wedge U^\vee$ of $\bigwedge^3 U^\vee$ contains no non-zero vectors of the form $f_1\wedge f_2\wedge f_3$ with $f_1,f_2,f_3\in U^\vee$, then $X\subset\mathbb{P}^{34}$ is a 3-dimensional smooth variety.*

*Moreover, each Mukai variety of the first species of dimension $\geq 3$ and sectional genus $\geq 6$, is projectively equivalent to a linear section of a variety $X$ constructed as above.*

*Remark* 2.9. A Mukai surface is of type $K3$. A Mukai variety of sectional genus $<6$ is a complete intersection of hypersurfaces.

We refer the reader to [Muk89] and [MM81] for the classification of Mukai varieties with $b_2\geq 2$. Finally we observe that in [Muk89] the classification is obtained under the assumption of the existence of a smooth divisor in $|\mathscr{O}_X(1)|$, a condition which is clearly satisfied in our case; in any event, this restriction has been removed in [Mel99].

## 2.2 *QEL/LQEL/CC*-varieties

In the following definition, we consider varieties having the simplest entry locus.

**Definition 2.10** ([Rus09, IR10, Zak93]). Let $X\subset\mathbb{P}^n$ be as in Assumption 1.1.

1. $X$ is called a *quadratic entry locus variety of type $\delta=\delta(X)$*, briefly a *QEL*-variety, if the general entry locus $\Sigma_p(X)$ is a quadric hypersurface (of dimension $\delta$).

2. $X$ is called a *locally quadratic entry locus variety of type* $\delta$, briefly an *LQEL*-variety, if each irreducible component of the general entry locus $\Sigma_p(X)$ is a quadric hypersurface; equivalently, $X$ is an *LQEL*-variety of type $\delta$ if through two general points there passes a quadric hypersurface of dimension $\delta$ contained in $X$.
3. $X$ is called a *conic-connected variety*, briefly a *CC*-variety, if through two general points of $X$ there passes an irreducible conic contained in $X$.

Of course, we have

$$X \text{ is a } QEL\text{-variety} \implies X \text{ is an } LQEL\text{-variety} \overset{\text{if } \delta\geq 0}{\implies} X \text{ is a } CC\text{-variety}$$

but the inverse implications are not true in general.

**Proposition 2.11.** *Let $X$ be smooth. Then*

1. *if $X' \subset \mathbb{P}^m$ (with $m \leq n$) is an isomorphic projection of $X \subset \mathbb{P}^n$, then $\delta(X') = \delta(X)$ and $X$ is an LQEL-variety (resp. CC-variety) if and only if $X'$ is an LQEL-variety (resp. CC-variety);*
2. *if $X$ is a QEL-variety (resp. LQEL-variety, CC-variety) of type $\delta(X) \geq 1$ and if $\widetilde{X} \subset \mathbb{P}^{n-1}$ is a general hyperplane section of $X \subset \mathbb{P}^n$, then $\widetilde{X}$ is a QEL-variety (resp. LQEL-variety, CC-variety) of type $\delta(\widetilde{X}) = \delta(X) - 1$.*

Now, for $X \subset \mathbb{P}^n$ as in Assumption 1.1 and with $\mathrm{Sec}(X) \subsetneq \mathbb{P}^n$, we consider the general tangential projection $\tau_{x,X} : X \dashrightarrow W_{x,X} \subset \mathbb{P}^{n-r-1}$ and we define the non-negative integer $\widetilde{\gamma}(X)$ as the dimension of the general fiber of the Gauss map

$$\mathscr{G}_{W_{x,X}} : W_{x,X} \dashrightarrow \mathbb{G}(r - \delta(X), n - r - 1).$$

From [Rus10, Lemma 2.3.4] we obtain the relation

$$\widetilde{\gamma}(X) = \gamma(X) - \delta(X). \tag{2.2}$$

Theorem 2.12 provides a sufficient condition for $X$ to be an *LQEL*-variety. For its proof, see [Rus10, Theorem 2.3.5].

**Theorem 2.12** (Scorza Lemma)**.** *Let $X \subset \mathbb{P}^n$ be as in Assumption 1.1, with* $\mathrm{Sec}(X) \subsetneq \mathbb{P}^n$ *and $\delta(X) \geq 1$ and let $x, y \in X$ ($x \neq y$) be general points. If $\widetilde{\gamma}(X) = 0$, then*

1. *The irreducible component of the closure of fiber of the rational map $\tau_{x,X} : X \dashrightarrow W_{x,X}$ passing through $y$ is either an irreducible quadric hypersurface of dimension $\delta(X)$ or a linear space of dimension $\delta(X)$, the last case occurring only for singular varieties.*
2. *There exists on $X \subset \mathbb{P}^n$ a $2(r - \delta(X))$-dimensional family $\mathscr{Q}$ of quadric hypersurfaces of dimension $\delta(X)$ such that there exists a unique member $Q_{x,y} \in \mathscr{Q}$ passing through the general points $x, y$. Furthermore, $Q_{x,y}$ is smooth at the points $x, y$ and it consists of the irreducible components of $\Sigma_p(X)$ passing through $x, y$, where $p \in \langle x, y \rangle$ is general.*

3. *If $X$ is smooth, then a general member of $\mathscr{Q}$ is smooth.*

Theorems 2.13 and 2.14 provide strong numerical and geometric constraints to smooth *LQEL*-variety of type $\delta \geq 3$.

**Theorem 2.13** ([Rus09])**.** *Let $X \subset \mathbb{P}^n$ be a smooth LQEL-variety of type $\delta(X) \geq 3$. Then*

1. *$X$ is a Fano variety of the first species of index $i(X) = \frac{\dim(X)+\delta(X)}{2}$.*
2. *If $x \in X$ is a general point, then $\mathscr{L}_{x,X} \subset \mathbb{P}^{\dim(X)-1}$ is a smooth irreducible QEL-variety of dimension $\dim(\mathscr{L}_{x,X}) = i(X) - 2$ and type $\delta(\mathscr{L}_{x,X}) = \delta(X) - 2$.*

As a direct consequence of Theorem 2.13, one obtains the following:

**Theorem 2.14** (Divisibility Theorem, [Rus09])**.** *If $X \subset \mathbb{P}^n$ is a smooth LQEL-variety of type $\delta(X) > 0$, then*

$$\dim(X) \equiv \delta(X) \bmod 2^{\lfloor(\delta(X)-1)/2\rfloor}.$$

Note that for a variety with secant defect $\delta = 0$, being an *LQEL*-variety imposes no restriction; differently, being a *QEL*-variety is fairly unique. Smooth *QEL*-varieties $X \subset \mathbb{P}^n$ of type $\delta = 0$ and with $\mathrm{Sec}(X) = \mathbb{P}^n$ are also called *varieties with one apparent double point*, briefly *OADP*-varieties (see [CMR04]). We have the following:

**Theorem 2.15** ([IR08, CMR04])**.** *If $X \subset \mathbb{P}^n$ is a smooth QEL-variety of type $\delta = 0$, then the general tangential projection $\tau_{x,X} : X \dashrightarrow W_{x,X}$ is birational.*

### 2.2.1 *LQEL*-varieties of higher type

The first part of Proposition 2.16 is an application of the definition of *LQEL*-variety (see [Rus09, Proposition 1.3]), while the second part follows from Theorem 2.13 (see [Rus09, Proposition 3.4]).

**Proposition 2.16.** *Let $X \subset \mathbb{P}^n$ be an LQEL-variety of type $\delta$ and dimension $r \geq 1$.*

1. *$\delta = r$ if and only if $X$ is a quadric hypersurface of $\mathbb{P}^n$.*
2. *If $X$ is smooth, then $\delta = r - 1$ if and only if $X$ is one of the following:*
   (a) *$\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ or one of its hyperplane sections;*
   (b) *the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or one of its isomorphic projections in $\mathbb{P}^4$.*

Theorem 2.17 is an application of Theorems 2.14, 2.7 and 2.8.

**Theorem 2.17** ([Rus09])**.** *Let $X \subset \mathbb{P}^n$ be a smooth $r$-dimensional LQEL-variety of type $\delta$. If $\frac{r}{2} < \delta < r$, then $X$ is projectively equivalent to one of the following:*

1. *the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;*
2. *a hyperplane section of the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$;*

3. *the Grassmannian* $\mathbb{G}(1,4) \subset \mathbb{P}^9$*;*
4. *a hyperplane section of the spinorial variety* $S^{10} \subset \mathbb{P}^{15}$*;*
5. *the spinorial variety* $S^{10} \subset \mathbb{P}^{15}$.

*Proof.* The proof is located in [Rus09, page 609], but we sketch it here. We have

$$0 < \frac{r-\delta}{2^{\lfloor(\delta-1)/2\rfloor}} < \frac{2\delta-\delta}{2^{\lfloor(\delta-1)/2\rfloor}} < 1, \quad \text{if } \delta \geq 9.$$

Thus, by Theorem 2.14, we obtain $\delta \leq 8$ and hence $r \leq 15$. Moreover, by Theorems 2.14 and 2.13 we obtain either that $(r,\delta) = (3,2)$ or $X$ is a Fano variety of the first species of coindex $c$ such that $(r,\delta,c) \in \{(5,3,2),\ (6,4,2),\ (9,5,3),\ (10,6,3)\}$. In the first case we conclude from Proposition 2.16; in the second case we conclude from Theorems 2.7 and 2.8. □

Theorem 2.18 is contained in [Rus09, Corollary 3.2]. It also classifies the so-called Severi varieties that we will treat in more detail in §2.3.

**Theorem 2.18** ([Rus09])**.** *Let* $X \subset \mathbb{P}^n$ *be a smooth linearly normal* $r$*-dimensional LQEL-variety of type* $\delta = r/2$*. Then* $X$ *is projectively equivalent to one of the following:*

1. *the cubic scroll* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(2)) \subset \mathbb{P}^4$*;*
2. *the Veronese surface* $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$*;*
3. *the Segre* 4*-fold* $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$*;*
4. *a* 4*-dimensional linear section of* $\mathbb{G}(1,4) \subset \mathbb{P}^9$*;*
5. *the Segre* 4*-fold* $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$*;*
6. *an* 8*-dimensional linear section of* $S^{10} \subset \mathbb{P}^{15}$*;*
7. *the Grassmannian* $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$*;*
8. *the Cartan variety* $E_6 \subset \mathbb{P}^{26}$*;*
9. *a* 16*-dimensional Fano variety* $X \subset \mathbb{P}^{25}$ *of coindex* 5*, with* $\mathrm{Sec}(X) = \mathbb{P}^{25}$ *and such that* $B_{x,X} \subset \mathbb{P}^{15}$ *is the union of a spinorial variety* $S^{10} \subset \mathbb{P}^{15}$ *with* $L_p(X) \simeq \mathbb{P}^7$*,* $p \in \mathbb{P}^{15} \setminus S^{10}$.

### 2.2.2 *CC*-varieties

From Proposition 2.11, the classification of the smooth *CC*-varieties is reduced to the case of linearly normal varieties. In this case, Theorem 2.19 reduces the classification to the study of Fano varieties having large index and Picard group $\mathbb{Z}$. Note that, conversely, such Fano varieties are *CC*-varieties (see [IR08]).

**Theorem 2.19** ([IR10])**.** *Let* $X \subset \mathbb{P}^n$ *be a smooth linearly normal CC-variety of dimension* $r$*. Then either* $X \subset \mathbb{P}^n$ *is a Fano variety of the first species of index* $i(X) \geq \frac{r+1}{2}$*, or* $X$ *is projectively equivalent to one of the following:*

1. *The Veronese embedding* $\nu_2(\mathbb{P}^r) \subset \mathbb{P}^{\frac{r(r+3)}{2}}$.

2. *The projection of* $\nu_2(\mathbb{P}^r)$ *from the linear space* $\langle \nu_2(\mathbb{P}^s)\rangle$, *where* $\mathbb{P}^s \subset \mathbb{P}^r$ *is a linear subspace; equivalently* $X \simeq \mathrm{Bl}_{\mathbb{P}^s}(\mathbb{P}^r)$ *embedded in* $\mathbb{P}^n$ *by the linear system of quadric hypersurfaces of* $\mathbb{P}^r$ *passing through* $\mathbb{P}^s$; *alternatively* $X \simeq \mathbb{P}_{\mathbb{P}^t}(\mathscr{E})$ *with* $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^t}(1)^{\oplus r-t} \oplus \mathscr{O}_{\mathbb{P}^t}(2)$, $t = 1,2,\ldots,r-1$, *embedded by* $|\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)|$. *Here* $n = \frac{r(r+3)}{2} - \binom{s+2}{2}$ *and s is an integer such that* $0 \leq s \leq r-2$.

3. *A hyperplane section of the Segre embedding* $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{n+1}$. *Here* $r \geq 3$ *and* $n = ab + a + b - 1$, *where* $a \geq 2$ *and* $b \geq 2$ *are such that* $a+b = r+1$.

4. *The Segre embedding* $\mathbb{P}^a \times \mathbb{P}^b \subset \mathbb{P}^{ab+a+b}$, *where* $a,b$ *are positive integers such that* $a+b = r$.

From Theorem 2.19 it follows that the smooth *CC*-varieties are Fano varieties and have second Betti number $b_2 \leq 2$; moreover those with $b_2 = 2$ are *LQEL*-varieties (see [IR10, Corollary 2.3]).

## 2.3 Severi varieties

Let $X \subset \mathbb{P}^n$ be as in Assumption 1.1 and further suppose that $\mathrm{Sec}(X) \subsetneq \mathbb{P}^n$. From [Rus10, Theorem 3.1.6] (see also [Zak93, II Corollary 2.11]) it follows that

$$n-1 \geq \dim(\mathrm{Sec}(X)) \geq \frac{3}{2}r + \frac{1-\dim(\mathrm{sing}(X))}{2} + \frac{\widetilde{\gamma}(X)}{2} \geq \frac{3}{2}r + \frac{1-\dim(\mathrm{sing}(X))}{2}; \tag{2.3}$$

in particular, if $X$ is also smooth, then

$$n \geq \frac{3}{2}r + 2. \tag{2.4}$$

**Definition 2.20.** A smooth variety $X$ as above is called a *Severi variety* if the inequality (2.4) is actually an equality.

Proposition 2.21 is contained in [Zak93, IV Theorem 2.4].

**Proposition 2.21** ([Zak93])**.** *Let* $X \subset \mathbb{P}^n$ *be a Severi variety.*

1. $\mathrm{Sec}(X)$ *is a normal hypersurface and* $\mathrm{sing}(\mathrm{Sec}(X)) = X$.

2. *For each* $p \in \mathrm{Sec}(X) \setminus X$, $\Sigma_p(X)$ *is a smooth quadric hypersurface of dimension* $r/2$.

3. *For each* $p \in \mathrm{Sec}(X) \setminus X$, *the linear projection* $\pi_L : X \dashrightarrow \mathbb{P}^r$ *from* $L = \langle \Sigma_p(X) \rangle$ *is birational and it is an isomorphism outside* $T_p(\mathrm{Sec}(X)) \cap X$.

### 2.3.1 Classification

Severi varieties are completely classified by Theorem 2.22; for its proof see [LV84] or [Zak93, IV Theorem 4.7] (see also [Rus10, §3.3]).

**Theorem 2.22.** *Each Severi variety $X$ is projectively equivalent to one of the following four varieties:*

1. *the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$;*
2. *the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;*
3. *the Grassmannian $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$;*
4. *the Cartan variety $E_6 \subset \mathbb{P}^{26}$.*

*All these varieties are homogeneous, rational and are defined by quadratic equations.*

There exists another interesting description of Severi varieties, given in Theorem 2.23. First recall that a quadro-quadric Cremona transformations is a birational transformations of $\mathbb{P}^n$ defined by quadratic forms and having inverse of the same kind; we also say that a Cremona transformation is special if its base locus is smooth and connected. Then we have the following:

**Theorem 2.23** ([ESB89], see also [PR11])**.** *Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a special Cremona transformation and let $\mathfrak{B}$ be its base locus. $\varphi$ is a quadro-quadric Cremona transformation if and only if $\mathfrak{B}$ is a Severi variety. Moreover, in this case $\varphi$ is an involution (i.e. $\varphi^{-1} = \varphi$).*

### 2.3.2 $R_1$-varieties

Let $X \subset \mathbb{P}^n$ be as in Assumption 1.1 and further suppose that $\mathrm{Sec}(X) \subsetneq \mathbb{P}^n$.

**Definition 2.24.** We say that $X$ enjoys the *$R_1$-property* (or briefly that $X$ is a *$R_1$-variety*) if for the general point $p \in \mathrm{Sec}(X)$ and the general hyperplane $H \in (\mathbb{P}^n)^*$ containing $T_p(\mathrm{Sec}(X))$ it holds that $H$ is $J$-tangent to $X$ along $\Gamma_p(X)$ (see Definitions 1.7 and 1.10).

Proposition 2.25 provides a simple criterion for the $R_1$-property; for its proof see [CC10, Proposition 5.13].

**Proposition 2.25.** *If $\widetilde{\gamma}(X) = 0$ and $X$ is different from a cone,*[1] *then $X$ is a $R_1$-variety.*

The importance of the $R_1$-property appears clear in the following:

**Theorem 2.26** ([CC10])**.** *If $X \subset \mathbb{P}^n$ is a $R_1$-variety with $n = \frac{3}{2}r + 2$, then $X$ is smooth (i.e. $X$ is a Severi variety).*

### 2.3.3 Equations of Severi varieties

Here we obtain the equations of the Severi varieties using a known construction (see [Zak93, IV Theorem 4.5]). As usual, we denote by $x_0, \ldots, x_n$ homogeneous coordinates on $\mathbb{P}^n$.

[1] Note that $X$ is different from a cone if and only if $\bigcap_{x \in X} T_x(X) = \emptyset$, see [Rus10, Proposition 1.2.6].

$\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$

The Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ can be defined by the $2 \times 2$ minors of the matrix (see [Har92]):

$$\begin{pmatrix} x_0 & x_3 & x_4 \\ x_3 & x_1 & x_5 \\ x_4 & x_5 & x_2 \end{pmatrix}$$

Its dimension is 2 and its degree is 4.

$\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$

The Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ can be defined by the $2 \times 2$ minors of the matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{pmatrix}$$

i.e. by the quadrics:

$$\begin{aligned} &x_5x_7 - x_4x_8, \quad x_2x_7 - x_1x_8, \quad x_5x_6 - x_3x_8, \\ &x_4x_6 - x_3x_7, \quad x_2x_6 - x_0x_8, \quad x_1x_6 - x_0x_7, \\ &x_2x_4 - x_1x_5, \quad x_2x_3 - x_0x_5, \quad x_1x_3 - x_0x_4. \end{aligned} \tag{2.5}$$

Its dimension is 4 and its degree is 6.

$\mathbb{G}(1,5) \hookrightarrow \mathbb{P}^{14}$

Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7$. This variety is defined by the $2 \times 2$ minors of the matrix:

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & x_7 \end{pmatrix} \tag{2.6}$$

Take the hyperplane $\mathbb{P}^7 = V(x_8)$ of $\mathbb{P}^8$ and consider the composition $\mathbb{P}^1 \times \mathbb{P}^3 \hookrightarrow \mathbb{P}^7 \hookrightarrow \mathbb{P}^8$. The image is defined by the 6 minors of the matrix (2.6) plus the 9 monomials $x_ix_8$, for $i = 0, \dots, 8$, and these 15 quadrics define a rational map $t_{\mathbb{P}^1 \times \mathbb{P}^3} : \mathbb{P}^8 \dashrightarrow \mathbb{P}^{14}$. The image of $t_{\mathbb{P}^1 \times \mathbb{P}^3}$ is the Grassmannian $\mathbb{G}(1,5)$ and it is defined by the 15 quadrics:

$$\begin{array}{cc} x_4x_6 - x_2x_7 + x_0x_9, & x_4x_{10} - x_2x_{11} + x_0x_{13}, \\ x_8x_{11} - x_7x_{12} + x_3x_{14}, & x_8x_{10} - x_6x_{12} + x_1x_{14}, \\ -x_3x_6 + x_1x_7 - x_0x_8, & -x_3x_{10} + x_1x_{11} - x_0x_{12}, \\ -x_2x_3 + x_1x_4 - x_0x_5, & -x_5x_6 + x_2x_8 - x_1x_9, \\ -x_7x_{10} + x_6x_{11} - x_0x_{14}, & x_5x_7 - x_4x_8 + x_3x_9, \\ x_5x_{10} - x_2x_{12} + x_1x_{13}, & x_9x_{11} - x_7x_{13} + x_4x_{14}, \\ x_5x_{11} - x_4x_{12} + x_3x_{13}, & -x_9x_{12} + x_8x_{13} - x_5x_{14}, \\ x_9x_{10} - x_6x_{13} + x_2x_{14}. & \end{array} \tag{2.7}$$

The dimension and the degree of $\mathbb{G}(1,5)$ respectively are 8 and 14.

$E_6 \hookrightarrow \mathbb{P}^{26}$

Consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. This variety is defined by the 3 quadrics:

$$-x_1x_3 + x_0x_4, \quad -x_2x_3 + x_0x_5, \quad -x_2x_4 + x_1x_5. \tag{2.8}$$

Take the hyperplane $\mathbb{P}^5 = V(x_6)$ of $\mathbb{P}^6$ and consider the composition $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5 \hookrightarrow \mathbb{P}^6$. The image is defined by the 3 quadrics (2.8) plus the 7 monomials $x_ix_6$, for $i = 0, \ldots, 6$, and these 10 quadrics define a rational map $t_{\mathbb{P}^1\times\mathbb{P}^2} : \mathbb{P}^6 \dashrightarrow \mathbb{P}^9$. The image of $t_{\mathbb{P}^1\times\mathbb{P}^2}$ is the Grassmannian $\mathbb{G}(1,4)$ and it is defined by the 5 quadrics:

$$\begin{array}{lll} x_4x_6 - x_3x_7 + x_0x_9, & -x_5x_7 + x_4x_8 - x_2x_9, & x_2x_6 - x_1x_7 + x_0x_8, \\ x_2x_3 - x_1x_4 + x_0x_5, & x_5x_6 - x_3x_8 + x_1x_9. & \end{array} \tag{2.9}$$

The dimension and the degree of $\mathbb{G}(1,4)$ respectively are 6 and 5. Now take the hyperplane $\mathbb{P}^9 = V(x_{10})$ of $\mathbb{P}^{10}$ and consider the composition $\mathbb{G}(1,4) \hookrightarrow \mathbb{P}^9 \hookrightarrow \mathbb{P}^{10}$. The image is defined by the 5 quadrics (2.9) plus the 11 monomials $x_ix_{10}$, for $i = 0, \ldots, 10$, and these 16 quadrics define a rational map $t_{\mathbb{G}(1,4)} : \mathbb{P}^{10} \dashrightarrow \mathbb{P}^{15}$. The image of $t_{\mathbb{G}(1,4)}$ is the spinorial variety $S^{10}$ and it is defined by the 10 quadrics:

$$\begin{array}{rr} x_7x_8 - x_6x_9 + x_5x_{10} - x_3x_{15}, & x_1x_6 + x_4x_7 - x_2x_{10} + x_3x_{13}, \\ -x_4x_5 + x_0x_6 - x_2x_8 + x_3x_{11}, & x_9x_{11} - x_8x_{12} + x_5x_{14} - x_0x_{15}, \\ x_{10}x_{12} - x_9x_{13} + x_7x_{14} + x_1x_{15}, & -x_1x_{11} - x_4x_{12} + x_0x_{13} - x_2x_{14}, \\ x_{10}x_{11} - x_8x_{13} + x_6x_{14} - x_4x_{15}, & x_1x_5 + x_0x_7 - x_2x_9 + x_3x_{12}, \\ -x_7x_{11} + x_6x_{12} - x_5x_{13} + x_2x_{15}, & x_1x_8 + x_4x_9 - x_0x_{10} + x_3x_{14}. \end{array} \tag{2.10}$$

The dimension and the degree of $S^{10}$ respectively are 10 and 12. Finally, take the hyperplane $\mathbb{P}^{15} = V(x_{16})$ of $\mathbb{P}^{16}$ and consider the composition $S^{10} \hookrightarrow \mathbb{P}^{15} \hookrightarrow \mathbb{P}^{16}$. The image is defined by the 10 quadrics (2.10) plus the 17 monomials $x_ix_{16}$, for $i = 0, \ldots, 16$, and these 27 quadrics define a rational map $t_{S^{10}} : \mathbb{P}^{16} \dashrightarrow \mathbb{P}^{26}$. The image of $t_{S^{10}}$ is the Cartan variety $E_6$ and it is defined by the 27 quadrics:

$$\begin{array}{rr} x_3x_{10} - x_{15}x_{17} + x_{14}x_{18} - x_{12}x_{21} + x_{11}x_{26}, & x_7x_{22} + x_5x_{23} + x_9x_{24} + x_2x_{25} - x_1x_{26}, \\ x_2x_{16} - x_1x_{18} + x_9x_{20} + x_0x_{22} - x_6x_{23}, & x_2x_{10} + x_{21}x_{22} - x_{15}x_{23} + x_{18}x_{24} - x_{20}x_{26}, \\ -x_2x_{13} + x_1x_{15} + x_5x_{20} + x_8x_{22} + x_6x_{24}, & x_1x_{10} - x_{19}x_{22} - x_{13}x_{23} + x_{16}x_{24} - x_{20}x_{25}, \\ -x_7x_{16} + x_1x_{17} - x_9x_{19} + x_4x_{23} + x_0x_{25}, & x_2x_{17} - x_7x_{18} + x_9x_{21} - x_3x_{23} + x_0x_{26}, \\ x_1x_{11} + x_0x_{13} + x_8x_{16} + x_6x_{19} - x_4x_{20}, & x_2x_{12} - x_9x_{15} - x_5x_{18} + x_3x_{22} - x_6x_{26}, \\ -x_7x_{12} + x_9x_{14} + x_5x_{17} + x_3x_{25} + x_4x_{26}, & -x_1x_{12} + x_9x_{13} + x_5x_{16} + x_4x_{22} + x_6x_{25}, \\ -x_5x_{11} - x_8x_{12} - x_3x_{13} + x_6x_{14} - x_4x_{15}, & x_2x_{19} - x_7x_{20} + x_1x_{21} + x_8x_{23} + x_0x_{24}, \\ x_7x_{10} - x_{14}x_{23} + x_{17}x_{24} - x_{21}x_{25} - x_{19}x_{26}, & x_0x_{10} - x_{18}x_{19} + x_{17}x_{20} - x_{16}x_{21} + x_{11}x_{23}, \\ x_8x_{10} + x_{15}x_{19} - x_{14}x_{20} + x_{13}x_{21} - x_{11}x_{24}, & x_1x_3 + x_2x_4 - x_0x_5 - x_6x_7 + x_8x_9, \\ -x_7x_{13} + x_1x_{14} + x_5x_{19} + x_4x_{24} - x_8x_{25}, & x_2x_{11} + x_0x_{15} + x_8x_{18} + x_3x_{20} - x_6x_{21}, \\ x_5x_{10} + x_{14}x_{22} + x_{12}x_{24} + x_{15}x_{25} - x_{13}x_{26}, & x_6x_{10} - x_{15}x_{16} + x_{13}x_{18} - x_{12}x_{20} + x_{11}x_{22}, \\ x_9x_{11} + x_0x_{12} - x_3x_{16} + x_6x_{17} - x_4x_{18}, & x_7x_{11} + x_0x_{14} + x_8x_{17} + x_3x_{19} - x_4x_{21}, \\ x_9x_{10} - x_{17}x_{22} - x_{12}x_{23} - x_{18}x_{25} + x_{16}x_{26}, & -x_2x_{14} + x_7x_{15} + x_5x_{21} + x_3x_{24} + x_8x_{26}, \\ -x_4x_{10} + x_{14}x_{16} - x_{13}x_{17} + x_{12}x_{19} + x_{11}x_{25}. & \end{array} \tag{2.11}$$

The dimension and the degree of $E_6$ respectively are 16 and 78.

### 2.3.4 $\mathscr{L}_{x,X}$ of Severi varieties

**Proposition 2.27.** *Table 2.1 describes the Hilbert scheme $\mathscr{L}_{x,X}$ of lines passing through a point $x$ of a Severi variety $X$.*

| $X$ | $\mathscr{L}_{x,X}$ |
|---|---|
| $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ | $\emptyset$ |
| $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ | $\mathbb{P}^1 \sqcup \mathbb{P}^1 \subset \mathbb{P}^3$ |
| $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$ | $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ |
| $E_6 \subset \mathbb{P}^{26}$ | $S^{10} \subset \mathbb{P}^{15}$ |

Table 2.1: $\mathscr{L}_{x,X}$ of Severi varieties.

*Proof.* Consider the rational maps above defined

$$t_{\mathbb{P}^1\times\mathbb{P}^3} : \mathbb{P}^8 \dashrightarrow \mathbb{P}^{14}, \quad t_{S^{10}} : \mathbb{P}^{16} \dashrightarrow \mathbb{P}^{26} \tag{2.12}$$

and similarly define

$$t_{\emptyset} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5, \quad t_{\mathbb{P}^1\sqcup\mathbb{P}^1} : \mathbb{P}^4 \dashrightarrow \mathbb{P}^8. \tag{2.13}$$

The rational maps in (2.12) and (2.13) are obviously birational onto them images and they have inverses defined by linear forms. The rest of the proof follows from Lemma 2.28 and the homogeneity of Severi varieties. □

**Lemma 2.28.** *Let $Y \subset H = \mathbb{P}^{r-1} \subset \mathbb{P}^r$ be the base locus of a quadratic birational map $t : \mathbb{P}^r \dashrightarrow \overline{t(\mathbb{P}^r)} = X \subset \mathbb{P}^n$ whose inverse is defined by linear forms. Let $x \in X$ be a general point and assume that $Y$ is reduced. Then*

$$\mathscr{L}_{x,X} \simeq Y \subset \mathbb{P}^{r-1}.$$

*Proof.* Let $p \in \mathbb{P}^r \setminus H$ be in the open set where $t$ is an isomorphism and put $x = t(p)$. Consider the isomorphism $H \xrightarrow{\simeq} \mathscr{L}_{p,\mathbb{P}^r}$ which sends the point $q \in H$ to the line $[\langle q, p\rangle] \in \mathscr{L}_{p,\mathbb{P}^r}$ and denote by $\mathscr{Y}$ the image of $Y$ in $\mathscr{L}_{p,\mathbb{P}^r}$. $\mathscr{Y}$ consists of all lines of $\mathbb{P}^r$ passing through $p$ and intersecting $Y$, so the images of these lines via $t$ are lines contained in $X$. Consider the closed subscheme

$$Z = \{(z,[l]) : z \in t(l)\} \hookrightarrow X \times \mathscr{Y}.$$

By universal property of $\mathscr{L}_{x,X}$, there exists a unique morphism $f : \mathscr{Y} \to \mathscr{L}_{x,X}$ such that we have the diagram (1.4) of fibred products. By the hypothesis on $t^{-1}$ and choice of the point $p$, we have a well-defined morphism

$$[l] \in \mathscr{L}_{x,X} \longmapsto t^{-1}(l) \in \mathscr{Y},$$

which is obviously the inverse of $f$. □

# Chapter 3

# Outline on Castelnuovo theory

This chapter is devoted to outline some well-known extensions of Castelnuovo's results and conjectures about them. These facts will be useful when we study the special quadratic birational transformations having base locus of small dimension, see Chap. 5 §5.5, Chap. 6 and App. B. The main references are: [Har82], [Cil87], [EGH93] and [Pet08] (see also the classical work [Cas89]).

**Notation and conventions** As always, we shall work over $\mathbb{C}$. Recall that, for any projective $\mathbb{C}$-scheme $\Lambda \subset \mathbb{P}^N$ the *Hilbert function* of $\Lambda$ is

$$h_\Lambda(t) = \dim_{\mathbb{C}} \left( \mathrm{Im} \left\{ \rho_t(\Lambda) : H^0(\mathscr{O}_{\mathbb{P}^N}(t)) \to H^0(\mathscr{O}_\Lambda(t)) \right\} \right),$$

where $t \in \mathbb{N}$ and $\rho_t(\Lambda)$ is the natural restriction map. So that for example $h_\Lambda(2)$ is the number of conditions imposed by $\Lambda$ on quadrics. A finite set of points $\Lambda \subset \mathbb{P}^N$ is in *uniform position* if for any subset $\Lambda' \subseteq \Lambda$ of order $\lambda'$ and for any $t \in \mathbb{N}$, it is $h_{\Lambda'}(t) = \min\{\lambda', h_\Lambda(t)\}$. If a set of points $\Lambda \subset \mathbb{P}^N$ is in uniform position, then it is also in *general position*, which means that any set of at most $N+1$ points in $\Lambda$ is linearly independent. If $\Lambda \subset \mathbb{P}^N$ is obtained by taking a general hyperplane section of a nondegenerate irreducible curve $C \subset \mathbb{P}^{N+1}$, then $\Lambda$ satisfies a stronger property than being in uniform position, namely $\Lambda$ is in *symmetric position* (see [Pet08]).

## 3.1 Castelnuovo's bound

Let $C \subset \mathbb{P}^{N+1}$, $N \geq 2$, be an irreducible nondegenerate projective curve of degree $\lambda$ and arithmetic genus $g$ and let $\Lambda \subset \mathbb{P}^N$ be a general hyperplane section of $C$. As it is well-known, there is an upper bound, the so-called *Castelnuovo's bound*, for $g$ as a function of $\lambda$ and $N$. Precisely one has the following:

**Proposition 3.1** (Castelnuovo's bound)**.** *Let notation be as above. Then*

$$g \leq \pi_0(\lambda, N) = \binom{q_0}{2} N + q_0 r_0, \tag{3.1}$$

*where*

$$\lambda - 1 = q_0 N + r_0, \quad 0 \leq r_0 < N.$$

*Proof.* We briefly review the idea of the proof. It is a standard fact (see e.g. [Cil87, Lemma 1.1]) that for $t \in \mathbb{N}$ one has:

$$h_\Lambda(t) \leq h_C(t) - h_C(t-1). \tag{3.2}$$

Now, the Hilbert polynomial of $C$ is $P_C(t) = \lambda t - g + 1$. So, by summing (3.2) over all $t \geq 1$, we get (see e.g. [Cil87, Lemma 1.4])

$$g \leq \sum_{t=1}^{\infty} (\lambda - h_\Lambda(t)). \tag{3.3}$$

Moreover, an easy consequence of the uniform position property of $\Lambda$ is that

$$h_\Lambda(t+s) \geq \min\{\lambda, h_\Lambda(t) + h_\Lambda(s) - 1\}, \quad \forall t,s \in \mathbb{N}, \tag{3.4}$$

from which it follows that

$$h_\Lambda(t) \geq \min\{\lambda, tN+1\}, \quad \forall t \in \mathbb{N}. \tag{3.5}$$

Then it is

$$\begin{array}{ll} h_\Lambda(t) \geq tN+1, & 1 \leq t \leq q_0 \\ h_\Lambda(t) = \lambda, & t \geq q_0 + 1 \end{array} \tag{3.6}$$

and (3.1) follows by using (3.3). □

More generally, (3.5) holds for a set of points $\Lambda$ in general position. For $t = 2$, this follows from the elementary Lemma 3.2; the general case follows from a similar argument.

**Lemma 3.2** (Usual Castelnuovo's argument)**.** *If $\Lambda \subset \mathbb{P}^N$ is a set of $\lambda \leq 2N+1$ points in general position, then $\Lambda$ imposes independent conditions to the quadrics of $\mathbb{P}^N$, i.e. $h_\Lambda(2) = \lambda$.*

*Proof.* We can assume $\lambda = 2N+1$. Put $\Lambda = \{p_0, \dots, p_{2N}\}$ and consider the hyperplanes $H_1 = \langle p_1, \dots, p_N \rangle$ and $H_2 = \langle p_{N+1}, \dots, p_{2N} \rangle$. Since the points are in general position, the quadric $H_1 \cup H_2$ contains the points $p_1, \dots, p_{2N}$, but not $p_0$. This proves the exactness of the sequence $0 \to H^0(\mathbb{P}^N, \mathscr{I}_{\Lambda,\mathbb{P}^N}(2)) \to H^0(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(2)) \to H^0(\Lambda, \mathscr{O}_\Lambda(2)) = \bigoplus_{i=0}^{2N} \mathbb{C} \to 0$, from which the assertion follows. □

The following well-known result describes $\Lambda$ in the case when $h_\Lambda(2)$ is minimal, assuming that $\lambda$ is not too small.

**Proposition 3.3** (Castelnuovo Lemma)**.** *Let $\Lambda \subset \mathbb{P}^N$ be a set of $\lambda \geq 2N+3$ points in general position. If $h_\Lambda(2) = 2N+1$, then $\Lambda$ lies on a rational normal curve of degree N, cut out by all quadrics containing $\Lambda$.*

By Proposition 3.3 is obtained the description of curves $C \subset \mathbb{P}^{N+1}$ that attain the maximal genus, the so-called *Castelnuovo curves*.

## 3.2 Refinements of Castelnuovo's bound

One question raised by Fano in [Fan94] is the following: *to get, under suitable conditions on $C$, better estimates for $h_\Lambda$ that (3.6), in order to obtain, using (3.3), better bounds for $g$.* If $\lambda \geq 2N+3$ and $h_\Lambda(2) = 2N+1$ then, by Proposition 3.3, no better estimates for $h_\Lambda$ can be found. So Fano's idea was to assume

$$h_\Lambda(2) \geq 2N+1+\vartheta, \quad 0 < \vartheta \leq \lambda - 2N - 1 \tag{3.7}$$

and to estimate $h_\Lambda$, and then $g$, under this hypothesis. The first result in this direction is the following:

**Proposition 3.4** (Fano)**.** *If (3.7) holds, then $g \leq \pi_0(\lambda - \vartheta, N) + \vartheta$ .*

*Proof.* See also [Cil87, Theorem 2.3]. Let $t \in \mathbb{N}$ and let $h = \lfloor n/2 \rfloor$. By using (3.4) and the hypothesis (3.7) we get

$$h_\Lambda(t) \geq \min\{\lambda, tN+1+h\vartheta\} \geq \min\{\lambda, tN+1+\vartheta\}.$$

So, by (3.3) we deduce the assertion. □

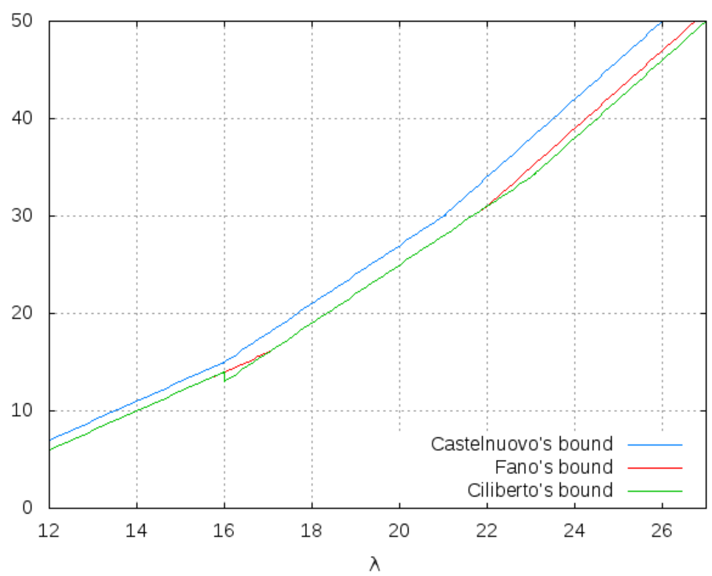


Figure 3.1: Comparison of upper bounds for $g$, when $N = 5$ and $\vartheta = 1$.

Fano's bound of Proposition 3.4 is too rough; in fact it can be attained only for $\lambda \leq 4N + 1 + \vartheta$. A more accurate result is the following (see [Cil87, Theorem 2.5]):

**Theorem 3.5** (Ciliberto)**.** *If (3.7) holds, then*

$$g \leq \theta(\lambda, N) = \begin{cases} \mu_\vartheta^2(2N+\vartheta) - \mu_\vartheta N + 2\mu_\vartheta\eta_\vartheta, & \text{if } \eta_\vartheta < N, \\ \mu_\vartheta^2(2N+\vartheta) - \mu_\vartheta N + 2\mu_\vartheta\eta_\vartheta + (\eta_\vartheta - N), & \text{if } \eta_\vartheta \geq N, \end{cases}$$

*where*

$$\lambda - 1 = \mu_\vartheta(2N+\vartheta) + \eta_\vartheta, \quad 0 \le \eta_\vartheta < 2N+\vartheta.$$

## 3.3 Eisenbud-Harris Conjecture

By extending Castelnuovo's lemma, one is naturally lead to make the following:

**Conjecture 3.6** (Eisenbud, Harris)**.** Let $\Lambda \subset \mathbb{P}^N$ be a finite set of $\lambda$ points in uniform position and let $\vartheta$ such that $0 \le \vartheta \le N-3$. If $\lambda \ge 2N+3+2\vartheta$, and $h_\Lambda(2) = 2N+1+\vartheta$, then $\Lambda$ lies on a curve $D \subset \mathbb{P}^N$ of degree at most $N+\vartheta$.

*Remark* 3.7. If the curve $D$ exists, then $D$ is unique and it is a component of $\mathrm{Bs}(|\mathscr{I}_\Lambda(2)|)$ (see e.g. [Pet08]).

*Remark* 3.8. The assumption $\vartheta \le N-3$ is necessary. For example, if $\vartheta = N-2$, then one could take $\Lambda$ to be a complete intersection of a del Pezzo surface of degree $N+1$ and two general quadrics to produce a counterexample.

Of course, the case $\vartheta = 0$ of Conjecture 3.6 is Castelnuovo's lemma; the case $\vartheta = 1$ is the following theorem due to Harris, see [Har82] (the result has been already know to Fano in [Fan94]).

**Theorem 3.9** (Fano, Harris)**.** *Let $\Lambda \subset \mathbb{P}^N$, $N \ge 3$, be a finite set of $\lambda \ge 2N+5$ points in uniform position such that $h_\Lambda(2) = 2N+2$. Then $\Lambda$ lies on an elliptic normal curve $D$ of degree $N+1$ in $\mathbb{P}^N$, cut out by all quadrics containing $\Lambda$.*

Cases $\vartheta = 2$ and $\vartheta = 3$ of Conjecture 3.6 have been proved by Petrakiev in [Pet08], under the stronger assumption that $\Lambda$ is a set of points in symmetric position (and hence also under the assumption that $\Lambda \subset \mathbb{P}^N$ is a general hyperplane section of a nondegenerate irreducible curve $C \subset \mathbb{P}^{N+1}$).

**Theorem 3.10** (Petrakiev)**.** *Let $\Lambda \subset \mathbb{P}^N$ be a finite set of $\lambda$ points in symmetric position.*

*(a) If $N \ge 4$, $h_\Lambda(2) = 2N+3$ and $\lambda \ge 2N+7$, then $\Lambda$ lies on a curve $D$ of degree $\le N+2$.*

*(b) If $N \ge 6$, $h_\Lambda(2) = 2N+4$ and $\lambda \ge 2N+9$, then $\Lambda$ lies on a curve $D$ of degree $\le N+3$.*

A results of same type as Theorem 3.10(a) was already known and due to Ciliberto (see [Cil87, Theorem 3.8]), but it is in general weaker than Theorem 3.10(a). The precise statement is as follows:

**Theorem 3.11** (Ciliberto)**.** *Let $\Lambda \subset \mathbb{P}^N$, $N \ge 5$, be a general hyperplane section of a nondegenerate irreducible curve $C \subset \mathbb{P}^{N+1}$. If $h_\Lambda(2) = 2N+3$ and $\lambda \ge \frac{8}{3}(N+1)$, then $\Lambda$ lies on an irreducible curve of degree $\le N+2$, cut out by all quadrics containing $\Lambda$.*

## 3.4 Bound on the degree of zero-dimensional quadratic schemes

Let $\Lambda \subset \mathbb{P}^N$ be a finite set of $\lambda$ points in uniform position and furthermore we hypothesize that $\Lambda$ is cut out by quadrics. Following [EGH93], we can ask: given $h_\Lambda(2)$, what is the largest possible $\lambda$? In other words, *What is the largest number $\lambda(h)$ of points of intersection of a linear system of quadrics of codimension h in the space of all quadrics in $\mathbb{P}^N$, given that the intersection of those quadrics is zero-dimensional?* In these terms, we may summarize the state of our knowledge in Table 3.1 and, by Conjecture 3.6, we can also extend the pattern, conjecturing

| $h_\Lambda(2)$ | $\lambda \leq$ | Proof |
|---|---|---|
| $N+1$ | $N+1$ | elementary |
| $N+2$ | $N+2$ | elementary |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $2N-1$ | $2N-1$ | elementary |
| $2N$ | $2N$ | elementary |
| $2N+1$ | $2N+2$ | Proposition 3.3 |
| $2N+2$ | $2N+4$ | Theorem 3.9 |
| $2N+3$ | $2N+6$ | Theorem 3.10(a) (if $N \geq 4$ and the points are in symmetric position) |
| $2N+4$ | $2N+8$ | Theorem 3.10(b) (if $N \geq 6$ and the points are in symmetric position) |

Table 3.1: State of our knowledge.

an upper bound on the largest $\lambda$, for $2N+5 \leq h_\Lambda(2) \leq 3N-2$. From the reasons given in [EGH93], this conjectured upper bound can be further extended. In order to present this, let $m = \binom{N+2}{2} - h_\Lambda(2)$ be the number of independent quadrics containing $\Lambda \subset \mathbb{P}^N$ and observe that, given $N$, any number $m \geq N+1$ can be uniquely written in the form

$$m = (N+1) + \binom{b}{2} + c, \quad b > c \geq 0.$$

With this notation, we make the following:

**Conjecture 3.12** (Eisenbud, Green, Harris)**.** If $\Lambda$ is any nondegenerate collection of $\lambda$ points in uniform position in $\mathbb{P}^N$ lying on $m$ independent quadrics whose intersection is zero-dimensional, then

$$\lambda \leq (2b-c+1)2^{N-b-1}.$$

*Remark* 3.13. The bound of Conjecture 3.12, if indeed it holds, is sharp: for $m$ quadrics, we can take $\Lambda$ the intersection of $N-b-1$ quadrics with a linearly normal variety of degree $2b-c+1$ and dimension $N-b-1$ in $\mathbb{P}^N$ (for example, the divisor residual to $c+1$ planes in the intersection of a rational normal $(N-b)$-fold scroll in $\mathbb{P}^N$ with a quadric).

| $\vartheta$ | $b$ | $c$ | $\lambda \leq$ |
|---|---|---|---|
| $-2 < \vartheta \leq N-4$ | $N-2$ | $N-\vartheta-4$ | $2N+2\vartheta+2$ |
| $N-4 < \vartheta \leq 2N-7$ | $N-3$ | $2N-\vartheta-7$ | $4\vartheta+8$ |
| $2N-7 < \vartheta \leq 3N-11$ | $N-4$ | $3N-\vartheta-11$ | $-8N+8\vartheta+32$ |
| $3N-11 < \vartheta \leq 4N-16$ | $N-5$ | $4N-\vartheta-16$ | $-32N+16\vartheta+112$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Table 3.2: Conjectured upper bound on $\lambda$.

Conjecture 3.12 is explained in Table 3.2, where $h_\Lambda(2) = 2N+1+\vartheta$, i.e. $m = N(N-1)/2-\vartheta$. Note that the first row of this table is a consequence of Conjecture 3.6. So, for $\vartheta = 0$ and $\vartheta = 1$, Conjecture 3.12 follows, respectively, from Proposition 3.3 and Theorem 3.9. For $\vartheta = 2$ and $\vartheta = 3$, it remains open in its generality, but it follows from Theorem 3.10 under the hypothesis that the points are in symmetric position. In [EGH93] another special case has been proved, precisely the case in which $m = N+1$ and $N \leq 6$.

# Chapter 4

# Introduction to quadratic birational transformations of a projective space into a quadric

In this chapter we begin the study of quadratic birational transformations $\varphi : \mathbb{P}^n \dashrightarrow \overline{\varphi(\mathbb{P}^n)} = \mathbf{S} \subset \mathbb{P}^{n+1}$, with particular regard to the case in which $\mathbf{S}$ is a quadric hypersurface.

## 4.1 Transformations of type $(2,1)$

**Definition 4.1.** A birational transformation $\varphi : \mathbb{P}^n \dashrightarrow \overline{\varphi(\mathbb{P}^n)} = \mathbf{S} \subset \mathbb{P}^{n+1}$ is said to be of type $(2,d)$ if it is quadratic (i.e. defined by a linear system of quadrics without fixed component) and $\varphi^{-1}$ can be defined by a linear system contained in $|\mathscr{O}_{\mathbf{S}}(d)|$, with $d$ minimal with this property. More generally, $\varphi$ is said to be of type $(d_1,d_2)$ if $\varphi$ (resp. $\varphi^{-1}$) is defined by a linear system contained in $|\mathscr{O}_{\mathbb{P}^n}(d_1)|$ (resp. $|\mathscr{O}_{\mathbf{S}}(d_2)|$), with $d_1$ and $d_2$ minimal. $\varphi$ is said to be *special* if its base locus is smooth and connected.

**Lemma 4.2.** *Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{S} \subset \mathbb{P}^{n+1}$ be a quadratic birational transformation, with base locus $\mathfrak{B}$ and image a normal nonlinear hypersurface $\mathbf{S}$. Then $h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) = n+2$.*

*Proof.* Resolve the indeterminacies of $\varphi$ with the diagram

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & \mathbf{S} \end{tikzcd} \tag{4.1}$$

where $\pi : X = \mathrm{Bl}_{\mathfrak{B}}(\mathbb{P}^n) \to \mathbb{P}^n$ is the blow-up of $\mathbb{P}^n$ along $\mathfrak{B}$, $E$ the exceptional divisor, $\pi' = \varphi \circ \pi$. By Zariski's Main Theorem ([Har77, III Corollary 11.4] or [Mum88, III §9]) we have $\pi'_*(\mathscr{O}_X) = \mathscr{O}_{\mathbf{S}}$ and by projection formula [Har77, II Exercise 5.1] it follows $\pi'_*(\pi'^*(\mathscr{O}_{\mathbf{S}}(1))) = \mathscr{O}_{\mathbf{S}}(1)$. Now, putting $V \subseteq H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(2))$ the linear vector space associated to the linear system $\sigma$ defining $\varphi$, we have the natural inclusions

$$\begin{array}{rcl} V & \hookrightarrow & H^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) \hookrightarrow H^0(X, \pi^*(\mathscr{O}_{\mathbb{P}^n}(2)) \otimes \pi^{-1}\mathscr{I}_{\mathfrak{B}} \cdot \mathscr{O}_X) \\ & \overset{\simeq}{\to} & H^0(X, \pi^*(\mathscr{O}_{\mathbb{P}^n}(2)) \otimes \mathscr{O}_X(-E)) \overset{\simeq}{\to} H^0(X, \pi'^*(\mathscr{O}_{\mathbf{S}}(1))) \\ & \overset{\simeq}{\to} & H^0(\mathbf{S}, \mathscr{O}_{\mathbf{S}}(1)) \overset{\simeq}{\to} H^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(1)). \end{array}$$

All these inclusions are isomorphisms, since $\dim(V) = n+2 = h^0(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}(1))$. □

**Proposition 4.3.** *Let $\varphi, \mathfrak{B}, \mathbf{S}$ be as in Lemma 4.2. The following conditions are equivalent:*

1. $h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(1)) \neq 0$*;*
2. $\mathfrak{B}$ *is a quadric of codimension* $2$*;*
3. $\mathfrak{B}$ *is a complete intersection;*
4. $\varphi$ *is of type* $(2,1)$.

*If one of the previous conditions is satisfied, then* $\mathbf{S}$ *is a quadric and* $\operatorname{rank}(\mathfrak{B}) = \operatorname{rank}(\mathbf{S}) - 2$.

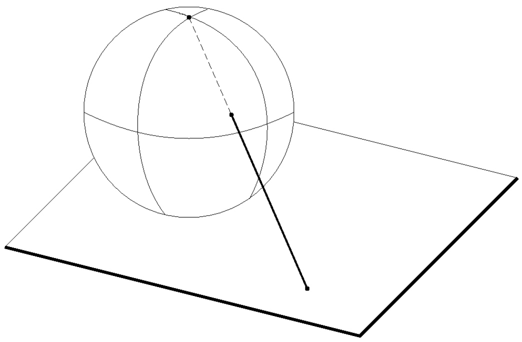

Figure 4.1: Stereographic projection.

*Proof.* ($1 \Rightarrow 2$ and $3$). If $f \in H^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(1))$ is a nonzero linear form, then the $n+1$ quadrics $x_0 f, \ldots, x_n f$ generate a subspace of codimension 1 of $H^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2))$, by Lemma 4.2. Thus there exists a quadric $F$ such that $H^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) = \langle F, x_0 f, \ldots, x_n f \rangle$ and $\mathfrak{B} = V(F, f) \subset V(f) \subset \mathbb{P}^n$.

($2 \Rightarrow 1$ and $4$). $\mathfrak{B}$ is necessarily degenerate and, modulo a change of coordinates on $\mathbb{P}^n$, we can suppose $\mathfrak{B} = V(x_0^2 + \cdots + x_s^2, x_n)$, with $s \leq n-1$. Hence

$$\varphi([x_0, \ldots, x_n]) = [x_0 x_n, \ldots, x_n^2, x_0^2 + \cdots + x_s^2] = [y_0, \ldots, y_{n+1}],$$

$$\mathbf{S} = V(y_0^2 + \cdots + y_s^2 - y_n y_{n+1}), \quad \varphi^{-1}([y_0, \ldots, y_{n+1}]) = [y_0, \ldots, y_n].$$

($4 \Rightarrow 2$). We can suppose that $\varphi^{-1}$ is the projection from the point $[0, \ldots, 0, 1]$. Thus, if $\varphi = (F_0, \ldots, F_{n+1})$, then $F_0, \ldots, F_n$ have a common factor $f$ and $\mathfrak{B} = V(F_{n+1}, f)$.

($3 \Rightarrow 1$). If $h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(1)) = 0$ and $\mathfrak{B}$ is a complete intersection, then every minimal system of generators of the ideal of $\mathfrak{B}$ consists of forms of degree 2, but then $h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) = n - \dim(\mathfrak{B}) < n+2$, absurd. □

## 4.2 Transformations of $\mathbb{P}^3$ and $\mathbb{P}^4$

In this section we shall keep the following:

**Assumption 4.4.** Let $\varphi : \mathbb{P}^n \dashrightarrow \overline{\varphi(\mathbb{P}^n)} = \mathbf{Q} \subset \mathbb{P}^{n+1}$ be a quadratic birational transformation into an irreducible (hence normal, by [Har77, I Exercise 5.12, II Exercise 6.5]) quadric hypersurface $\mathbf{Q}$ and moreover suppose that its base locus $\emptyset \neq \mathfrak{B} \subset \mathbb{P}^n$ is reduced.

**Lemma 4.5.** *Let $X \subseteq \mathfrak{B}$ be a degenerate irreducible component of $\mathfrak{B}$.*

1. *If* $\operatorname{codim}_{\mathbb{P}^n}(X) = 2$ *then* $\deg(X) \leq 2$ *and, if* $\deg(X) = 2$ *then* $X = \mathfrak{B}$ *and* $\mathfrak{B}$ *is a quadric.*
2. *If* $\operatorname{codim}_{\mathbb{P}^n}(X) = 3$ *then* $\deg(X) \leq 4$ *and, if* $\deg(X) = 4$ *then we have* $h^0(\mathbb{P}^n, \mathscr{I}_X(2)) = h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) + 1$.

*Proof.* We can choose coordinates $x_0, \ldots, x_n$ on $\mathbb{P}^n$ such that $X \subset V(x_n) \subset \mathbb{P}^n$ and we consider the restriction map $u : H^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) \to H^0(\mathbb{P}^{n-1}, \mathscr{I}_X(2))$, defined by $u(F(x_0, \ldots, x_n)) = F(x_0, \ldots, x_{n-1}, 0)$. Note that if $F \in H^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2))$, then $F \in \langle u(F) \rangle \oplus \langle x_0 x_n, \ldots, x_{n-1} x_n, x_n^2 \rangle$ and, in particular, $H^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) \subseteq \operatorname{Im}(u) \oplus \langle x_0 x_n, \ldots, x_{n-1} x_n, x_n^2 \rangle$. Thus $\dim(\operatorname{Im}(u)) \geq 1$ and, if $\dim(\operatorname{Im}(u)) = 1$ then $\mathfrak{B}$ is a quadric hypersurface in $\mathbb{P}^{n-1}$. Now suppose $\operatorname{codim}_{\mathbb{P}^n}(X) = 2$. From the above, there exists $\bar{F} \in H^0(\mathbb{P}^{n-1}, \mathscr{I}_X(2))$ and $X$ has to be an irreducible component of $V(\bar{F})$. It follows that $\deg(X) \leq \deg(\bar{F}) = 2$ and, if $\deg(X) = 2$ then $X = V(\bar{F})$ and $h^0(\mathbb{P}^{n-1}, \mathscr{I}_X(2)) = \dim(\operatorname{Im}(u)) = 1$. Suppose $\operatorname{codim}_{\mathbb{P}^n}(X) = 3$. From the above, there exist $\bar{F}, \bar{F}' \in H^0(\mathbb{P}^{n-1}, \mathscr{I}_X(2))$ which are linearly independent and $X$ has to be contained in the complete intersection $V(\bar{F}, \bar{F}')$. It follows that $\deg(X) \leq 4$ and, if $\deg(X) = 4$ then we have $h^0(\mathbb{P}^{n-1}, \mathscr{I}_X(2)) = \dim(\operatorname{Im}(u)) = 2$. □

**Proposition 4.6.** *If $n = 3$, then either*

*(i)* *$\varphi$ is of type $(2,1)$, or*

*(ii)* *$\varphi$ is of type $(2,2)$,* $\operatorname{rank}(\mathbf{Q}) = 4$ *and $\mathfrak{B}$ is the union of a line $r$ with two points $p_1, p_2$ such that $\langle p_1, p_2 \rangle \cap r = \emptyset$.*

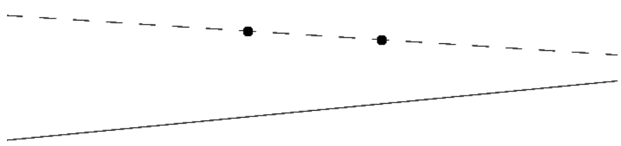

Figure 4.2: Base locus when $n = 3$.

*Proof.* Since $\operatorname{codim}_{\mathbb{P}^3}(\mathfrak{B}) \geq 2$, $\mathfrak{B}$ has the following decomposition into irreducible components:

$$\mathfrak{B} = \bigcup_j C_j \cup \bigcup_i p_i,$$

where $C_j$ and $p_i$ are respectively curves and points, with $\deg(C_j) \leq 3$, by Bézout's Theorem. If one of $C_j$ is nondegenerate, then $\deg(C_j) = 3$ and $C_j$ is the twisted cubic curve, by [Har92, Proposition 18.9]. This would produce the absurd result that $5 = h^0(\mathbb{P}^3, \mathscr{I}_{\mathfrak{B},\mathbb{P}^3}(2)) \leq h^0(\mathbb{P}^3, \mathscr{I}_{C_j,\mathbb{P}^3}(2)) = 3$ and hence we have that every $C_j$ is degenerate. By Lemma 4.5, using the fact that a line imposes 3 conditions to the quadrics, it follows that $\mathfrak{B}$ is a (irreducible or not) conic or as asserted in (ii). In the latter case, modulo a change of coordinates, we can suppose $r = V(x_2, x_3)$, $p_1 = [0,0,1,0]$, $p_2 = [0,0,0,1]$ and we get

$$\mathfrak{B} = V(x_0x_2,\ x_0x_3,\ x_1x_2,\ x_1x_3,\ x_2x_3), \quad \mathbf{Q} = V(y_0y_3 - y_1y_2). \tag{4.2}$$

□

**Lemma 4.7.** *If $n \geq 3$ and $\mathfrak{B}$ has an isolated point, then* $\operatorname{rank}(\mathbf{Q}) \leq 4$.

*Proof.* Let $p$ be an isolated (reduced) point of $\mathfrak{B}$ and consider the diagram

$$\begin{tikzcd}
\mathbb{P}^{n-1} \simeq E \arrow[r, hook] \arrow[drr, dashed, bend left, "\bar{\varphi}"] & \mathrm{Bl}_p(\mathbb{P}^n) \arrow[d, "\pi"] \arrow[dr, dashed] & \\
& \mathbb{P}^n \arrow[r, dashed, "\varphi"] & \mathbf{Q} \subset \mathbb{P}^{n+1}
\end{tikzcd}$$

By the hypothesis on $p$, $\bar{\varphi}$ is a linear morphism and so the quadric $\mathbf{Q} = \mathbf{Q}^n$ contains a $\mathbb{P}^{n-1}$. This, since $n \geq 3$, implies $\operatorname{rank}(\mathbf{Q}) \leq 4$. □

**Proposition 4.8.** *If $n = 4$ and* $\operatorname{rank}(\mathbf{Q}) \geq 5$*, then either*

*(i) $\varphi$ is of type $(2,1)$, or*

*(ii) $\varphi$ is of type $(2,2)$, $\mathbf{Q}$ is smooth and $\mathfrak{B}$ is one of the following (see Figure 4.3):*

*(a) the rational normal quartic curve,*

*(b) the union of the twisted cubic curve in a hyperplane $H \subset \mathbb{P}^4$ with a line not contained in $H$ and intersecting the twisted curve,*

*(c) the union of an irreducible conic with two skew lines that intersect it,*

*(d) the union of three skew lines with another line that intersects them.*

*Proof.* By Lemma 4.7 and since $\operatorname{codim}_{\mathbb{P}^4}(\mathfrak{B}) \geq 2$, $\mathfrak{B}$ has the following decomposition into irreducible components:

$$\mathfrak{B} = \bigcup_i S_i \cup \bigcup_j C_j,$$

where $S_i$ and $C_j$ are respectively surfaces and curves. Let $S$ be one of $S_i$ and let $C$ be one of $C_j$. We discuss all possible cases.

*Case* 4.8.1 ($\deg(S) \geq 4$). This case is impossible by Bézout's Theorem.

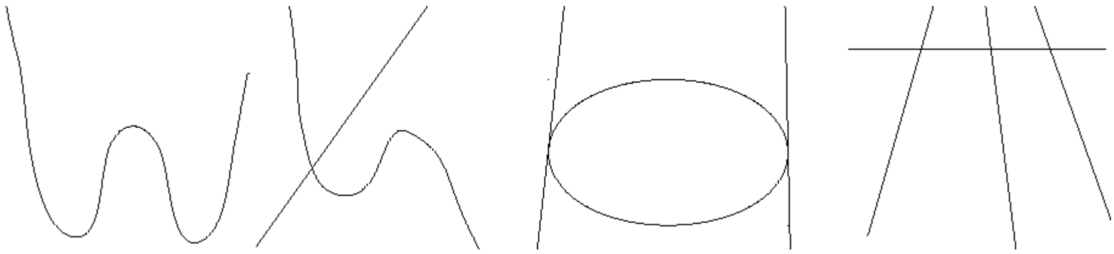

Figure 4.3: Base loci when $n = 4$.

*Case* 4.8.2 ($\deg(S) = 3$, $S$ nondegenerate). Cutting $S$ with a general hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$, we obtain the twisted cubic curve $\Gamma \subset \mathbb{P}^3$. Hence, from the exact sequence $0 \to \mathscr{I}_{S,\mathbb{P}^4}(-1) \to \mathscr{I}_{S,\mathbb{P}^4} \to \mathscr{I}_{\Gamma,\mathbb{P}^3} \to 0$, we deduce the contradiction $6 = h^0(\mathbb{P}^4, \mathscr{I}_{\mathfrak{B},\mathbb{P}^4}(2)) \leq h^0(\mathbb{P}^4, \mathscr{I}_{S,\mathbb{P}^4}(2)) \leq h^0(\mathbb{P}^3, \mathscr{I}_{\Gamma,\mathbb{P}^3}(2)) = 3$.

*Case* 4.8.3 ($\deg(S) \geq 2$, $S$ degenerate). By Lemma 4.5 we have $\deg(S) = 2$ and $S = \mathfrak{B}$ is a quadric.

*Case* 4.8.4 ($\deg(C) \geq 5$, $C$ nondegenerate). Take a general hyperplane $\mathbb{P}^3 \subset \mathbb{P}^4$ and put $\Lambda = \mathbb{P}^3 \cap C$. $\Lambda$ is a set of $\lambda \geq 5$ points of $\mathbb{P}^3$ in general position and therefore, by Lemma 3.2, we get the contradiction

$$6 = h^0(\mathbb{P}^4, \mathscr{I}_{\mathfrak{B}}(2)) \leq h^0(\mathbb{P}^4, \mathscr{I}_C(2)) \leq h^0(\mathbb{P}^3, \mathscr{I}_\Lambda(2)) \leq \begin{cases} 10 - \lambda \leq 5, & \text{if } \lambda \leq 7 \\ 10 - 7 = 3, & \text{if } \lambda \geq 7. \end{cases}$$

*Case* 4.8.5 ($\deg(C) \geq 5$, $C$ degenerate). This is impossible by Lemma 4.5.

*Case* 4.8.6 ($\deg(C) = 4$, $C$ nondegenerate). By [Har92, Proposition 18.9], $C$ is the rational normal quartic curve and one of its parameterizations is $[s,t] \in \mathbb{P}^1 \mapsto [s^4, s^3t, s^2t^2, st^3, t^4] \in \mathbb{P}^4$. We have $h^0(\mathbb{P}^4, \mathscr{I}_{C,\mathbb{P}^4}(2)) = 6$ and hence $C = \mathfrak{B}$ and

$$\begin{aligned} \mathfrak{B} &= V(x_2^2 - x_1x_3,\ x_2x_3 - x_1x_4,\ x_0x_4 - x_1x_3,\ x_3^2 - x_2x_4,\ x_0x_2 - x_1^2,\ x_0x_3 - x_1x_2), && (4.3) \\ \mathbf{Q} &= V(y_0y_2 - y_1y_5 + y_3y_4). && (4.4) \end{aligned}$$

*Case* 4.8.7 ($\deg(C) = 4$, $C$ degenerate). This case is impossible by Lemma 4.5 and Proposition 4.3.

*Case* 4.8.8 ($\deg(C) = 3$, $\langle C \rangle = \mathbb{P}^3$). Modulo a change of coordinates, $C$ is the twisted cubic curve parameterized by $[s,t] \in \mathbb{P}^1 \mapsto [s^3, s^2t, st^2, t^3, 0] \in V(x_4)$ and, by the reduction obtained, since $h^0(\mathbb{P}^4, \mathscr{I}_C(2)) = h^0(\mathbb{P}^4, \mathscr{I}_{\mathfrak{B}}(2)) + 2$, it follows $\mathfrak{B} = C \cup r$, where $r$ is a line that intersects $C$ in a single point transversely. We can choose the intersection point to be $p = [1,0,0,0,0]$ and then we have $r = \{[s + q_0t, q_1t, q_2t, q_3t, q_4t] : [s,t] \in \mathbb{P}^1\}$, for some $q = [q_0, q_1, q_2, q_3, q_4] \in \mathbb{P}^4$. By Proposition 4.3 we have $q_4 \neq 0$ and only for simplicity of notation we take $q = [0,0,0,0,1]$, hence $r = V(x_1, x_2, x_3)$. So we obtain

$$\begin{aligned} \mathfrak{B} &= V(x_1^2 - x_0x_2,\ x_3x_4,\ x_0x_3 - x_1x_2,\ x_2x_4,\ x_1x_3 - x_2^2,\ x_1x_4), && (4.5) \\ \mathbf{Q} &= V(-y_4y_5 + y_2y_3 + y_0y_1). && (4.6) \end{aligned}$$

*Case* 4.8.9 ($\deg(C) = 3$, $\langle C \rangle = \mathbb{P}^2$). It is impossible because otherwise $\mathfrak{B}$ would contain the entire plane spanned by $C$.

*Case* 4.8.10 ($\deg(S) = 1$). Since planes, conics and lines impose to the quadrics respectively 6, 5 and 3 conditions, we have two subcases:

*Subcase* 4.8.10.1 ($\mathfrak{B} = S \cup \Gamma$, $\Gamma$ conic, $\#(\Gamma \cap S) = 2$). This is impossible because otherwise there exists a line cutting $\mathfrak{B}$ in exactly 3 points.

*Subcase* 4.8.10.2 ($\mathfrak{B} = S \sqcup l$, $l$ line). This contradicts the birationality of $\varphi$.

*Case* 4.8.11 ($\deg(C) = 2$, $C$ irreducible). Let $C', l, l'$ be respectively eventual conic and eventual lines contained in $\mathfrak{B}$. We discuss the subcases:

*Subcase* 4.8.11.1 ($\mathfrak{B} = C \cup C'$, $\#(C \cap C') = 1$). In this case, we would have that $\mathrm{rank}(\mathbf{Q}) = 4$, against the hypothesis.

*Subcase* 4.8.11.2 ($\mathfrak{B} = C \cup C' \cup l$, $\#(C \cap C') = 2$, $\#(C \cap l) = \#(C' \cap l) = 1$). $\mathfrak{B} \supseteq C \cup C' \cup l$ implies that $\mathfrak{B}$ is a quadric of codimension 2.

*Subcase* 4.8.11.3 ($\mathfrak{B} = C \cup l \cup l'$, $\#(l \cap l') = 0$, $\#(C \cap l) = \#(C \cap l') = 1$). This case is really possible and, modulo a change of coordinates, we have

$$\mathfrak{B} = V(x_2^2 + x_0x_1,\ x_3x_4,\ x_1x_3,\ x_2x_4,\ x_2x_3,\ x_0x_4), \tag{4.7}$$
$$\mathbf{Q} = V(-y_2y_5 - y_3y_4 + y_0y_1). \tag{4.8}$$

*Case* 4.8.12 ($\deg(C) = 2$, $C$ reducible). By the reduction obtained, $\mathfrak{B}$ contains two other skew lines, each of which intersects $C$ at a single point. This case is really possible and, modulo a change of coordinates, we have

$$\mathfrak{B} = V(x_1x_2,\ x_3x_4,\ x_0x_3,\ x_2x_4,\ x_2x_3,\ x_0x_4 - x_1x_4), \tag{4.9}$$
$$\mathbf{Q} = V(-y_4y_5 + y_2y_3 - y_0y_1). \tag{4.10}$$

□

Example 4.9 suggests that a possible generalization of Proposition 4.8 to the case where $\mathfrak{B}$ in nonreduced and $rk(\mathbf{Q}) < 5$ may not be trivial.

**Example 4.9.** The rational map $\varphi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^5$ defined by

$$\varphi([x_0,x_1,x_2,x_3,x_4]) = [x_0^2, -x_0x_1, -x_0x_2, x_1^2 - x_0x_3, 2x_1x_2 - x_0x_4, x_2^2],$$

is birational into its image, which is the quadric of rank 3, $\mathbf{Q} = V(y_0y_5 - y_2^2)$. If $\pi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ is the projection from the point $[0,0,0,0,0,1]$, then the composition $\pi \circ \varphi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ is an involution. The base locus $\mathfrak{B}$ of $\varphi$ is everywhere nonreduced, $(\mathfrak{B})_{\mathrm{red}} = V(x_0, x_1, x_2)$ and $P_{\mathfrak{B}}(t) = 4t + 1$.

# Chapter 5

# On special quadratic birational transformations of a projective space into a hypersurface

Consider a special birational transformation $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{S} \subset \mathbb{P}^{n+1}$ of type $(2,d)$ from a complex projective space into a nonlinear and sufficiently regular hypersurface $\mathbf{S}$. The blow-up $\pi : X = \mathrm{Bl}_{\mathfrak{B}}(\mathbb{P}^n) \to \mathbb{P}^n$ of $\mathbb{P}^n$ along the base locus $\mathfrak{B}$ resolves the indeterminacies of the transformation $\varphi$. So, comparing the two ways in which it is possible to write the canonical class $K_X$, with respect to $\pi$ and $\pi' := \varphi \circ \pi$, we get a formula expressing the dimension of $\mathfrak{B}$ as a function of $n$, $d$ and $\deg(\mathbf{S})$, see Proposition 5.2. The primary advantage in dealing with the case of quadratic transformations is that $\mathfrak{B}$ is a *QEL*-variety (see Definition 2.10). Therefore it is possible to apply the main results of the theory of *QEL*-varieties; in particular, the Divisibility Theorem (Theorem 2.14), together with the formula on the dimension of $\mathfrak{B}$, drastically reduces the set of quadruples $(\dim(\mathfrak{B}), n, d, \deg(\mathbf{S}))$ for which such a $\varphi$ exists.

The $\varphi$ of type $(2,1)$ have been described in Proposition 4.3 as the only transformations whose base locus is a quadric of codimension 2; in particular, modulo projective transformations, there is only one example. With regard to the $\varphi$'s of type $(2,2)$ into a quadric hypersurface $\mathbf{Q}$, it is known that a special Cremona transformation $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n+1}$ of type $(2,2)$ has as base locus a Severi variety (see Theorem 2.23) and moreover, modulo projective transformations, there exist only four Severi varieties (see Theorem 2.22). Now, if we restrict these transformations to a general hyperplane $\mathbb{P}^n \subset \mathbb{P}^{n+1}$, we clearly obtain special transformations $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{Q} \subset \mathbb{P}^{n+1}$ of type $(2,2)$ (see Example 5.13). In Theorem 5.19 we prove that all examples of $\varphi$ of type $(2,2)$ into a quadric $\mathbf{Q}$ arise in this way; in particular, their base loci are hyperplane sections of Severi varieties. Regarding special transformations of type $(2,2)$ into a cubic and quartic hypersurface we are able to determine some invariants of the base locus as the Hilbert polynomial and the Hilbert scheme of lines passing through a point (Propositions 5.30 and 5.31).

Another approach to the study of all $\varphi$ of type $(2,d)$ is their classification according to the dimension of the base locus. In Table 5.5 (which is constructed via Propositions 5.21 and 5.25), we provide a list of all possible base loci when the dimension is at most 3, although in one case we do not know if it really exists. As a consequence, in Corollary 5.28, we obtain that a special transformation $\varphi$ of type $(2,3)$ into a cubic hypersurface $\mathbf{S}$ has as base locus the blow-up $\mathrm{Bl}_{\{p_1,\ldots,p_5\}}(Q)$ of 5 points $p_1,\ldots,p_5$ in a smooth quadric $Q\subset\mathbb{P}^4$, embedded in $\mathbb{P}^8$ by the linear system $|2H_{\mathbb{P}^4}|_Q-p_1-\cdots-p_5|$ (see also Example 5.17).

Throughout the chapter, unless otherwise specified, we shall keep the following:

**Assumption 5.1.** Let $n\geq 3$ and $\varphi:\mathbb{P}^n\dashrightarrow\overline{\varphi(\mathbb{P}^n)}=\mathbf{S}\subset\mathbb{P}^{n+1}$ be a special birational transformation of type $(2,d)$, with $d\geq 2$ and with $\mathbf{S}$ a factorial hypersurface of degree $\Delta\geq 2$ (in particular, when $\Delta=2$, it is enough to require $\mathrm{rank}(\mathbf{S})\geq 5$). Observe that $\mathrm{Pic}(\mathbf{S})=\mathbb{Z}\langle\mathscr{O}_{\mathbf{S}}(1)\rangle$ (see [Har70, IV Corollary 3.2]) and $\omega_{\mathrm{reg}(\mathbf{S})}\simeq\mathscr{O}_{\mathrm{reg}(\mathbf{S})}(\Delta-n-2)$. Moreover, denote by $\mathfrak{B}\subset\mathbb{P}^n$ and $\mathfrak{B}'\subset\mathbf{S}\subset\mathbb{P}^{n+1}$ respectively the base locus of $\varphi$ and $\varphi^{-1}$ and assume[1]

$$(\mathfrak{B}')_{\mathrm{red}}\neq(\mathrm{sing}(\mathbf{S}))_{\mathrm{red}}. \tag{5.1}$$

Put $r=\dim(\mathfrak{B})$, $r'=\dim(\mathfrak{B}')$, $\delta=\delta(\mathfrak{B})$ the secant defect, $\lambda=\deg(\mathfrak{B})$, $g=g(\mathfrak{B})$ the sectional genus, $P_{\mathfrak{B}}(t)$ the Hilbert polynomial, $i(\mathfrak{B})$ and $c(\mathfrak{B})$ respectively index and coindex (when $\mathfrak{B}$ is a Fano variety).

## 5.1 Properties of the base locus

**Proposition 5.2.**

1. *$\mathfrak{B}$ is a QEL-variety of dimension and type given by*

$$r=\frac{dn-\Delta-3d+3}{2d-1},\quad \delta=\frac{n-2\Delta-2d+4}{2d-1}.$$

2. *$\mathfrak{B}'$ is irreducible, generically reduced, of dimension*

$$r'=\frac{2(dn-n+\Delta-d-1)}{2d-1}.$$

*Proof.* See also [ESB89, Propositions 2.1 and 2.3]. Consider the diagram (4.1), where $\pi:X=\mathrm{Bl}_{\mathfrak{B}}(\mathbb{P}^n)\to\mathbb{P}^n$ and $\pi'=\varphi\circ\pi$. $X$ can be identified with the graphic $\mathrm{Graph}(\varphi)\subset\mathbb{P}^n\times\mathbf{S}$ and the maps $\pi$, $\pi'$ can be identified with the projections onto the factors. It follows that $(\mathfrak{B})_{\mathrm{red}}$ (resp. $(\mathfrak{B}')_{\mathrm{red}}$) is the set of the points $x$ such that the fiber $\pi^{-1}(x)$ (resp. ${\pi'}^{-1}(x)$) has positive dimension. Denote by $E$ the exceptional divisor of $\pi$, $E'={\pi'}^{-1}(\mathfrak{B}')$, $H\in|\pi^*(\mathscr{O}_{\mathbb{P}^n}(1))|$, $H'\in|{\pi'}^*(\mathscr{O}_{\mathbf{S}}(1))|$ and note that, by the factoriality of $\mathbf{S}$ and by the proof of [ESB89, Proposition 1.3], it follows that $E'$ is an irreducible divisor, in particular $\mathfrak{B}'$ is irreducible. Moreover we have the relations

$$H\sim dH'-E',\quad H'\sim 2H-E, \tag{5.2}$$

from which we get

$$E\sim(2d-1)H'-2E',\quad E'\sim(2d-1)H-dE, \tag{5.3}$$

[1] See Example 5.18 for an explicit example of special quadratic birational transformation for which (5.1) is not satisfied.

and in particular by (5.3) it follows $E' = (E')_{\mathrm{red}}$. Put

$$U = \mathrm{reg}(\mathbf{S}) \setminus \mathrm{sing}\left((\mathfrak{B}')_{\mathrm{red}}\right), \quad V = \pi'^{-1}(U), \quad Z = U \cap (\mathfrak{B}')_{\mathrm{red}}. \tag{5.4}$$

Observe that, since $X$ is smooth and we have assumed (5.1), we have $Z \neq \emptyset$. Thus, by [ESB89, Theorem 1.1], $\pi'|_V : V \to U$ coincides with the blow-up of $U$ along $Z$; in particular

$$\mathrm{Pic}(X) = \mathbb{Z}\langle H\rangle \oplus \mathbb{Z}\langle E\rangle = \mathbb{Z}\langle H'\rangle \oplus \mathbb{Z}\langle E'\rangle = \mathrm{Pic}(V).$$

Now, by [Har77, II Exercise 8.5] and (5.2) and (5.3), we have

$$\begin{aligned} K_X &\sim \pi^*(K_{\mathbb{P}^n}) + (\mathrm{codim}_{\mathbb{P}^n}(\mathfrak{B}) - 1)\,E \\ &\sim (-n-1)\,H + (n-r-1)\,E \\ &\sim (-2dr + r + dn - n - 3d + 1)\,H' + (2r - n + 3)\,E' \end{aligned} \tag{5.5}$$

and also

$$\begin{aligned} K_X|V \sim K_V &\sim \pi'^*(K_{\mathrm{reg}(\mathbf{S})}|U) + (\mathrm{codim}_U(Z) - 1)\,E' \\ &\sim (\deg(\mathbf{S}) - (n+1) - 1)\,H' + (\mathrm{codim}_{\mathbf{S}}(\mathfrak{B}') - 1)\,E' \\ &\sim (\Delta - n - 2)\,H' + (n - r' - 1)\,E'. \end{aligned} \tag{5.6}$$

By the comparison of (5.5) and (5.6) we obtain the expressions of $r = \dim(\mathfrak{B})$ and $r' = \dim(\mathfrak{B}')$. Moreover, the proof of [ESB89, Proposition 2.3(a),(b)] adapts to our case, producing that the secant variety $\mathrm{Sec}(\mathfrak{B})$ is a hypersurface of degree $2d-1$ in $\mathbb{P}^n$ and $\mathfrak{B}$ is a *QEL*-variety of type $\delta = n - r' - 2$. Finally, we have $\mathfrak{B}' \cap U = (\mathfrak{B}')_{\mathrm{red}} \cap U$ by the argument in [ESB89, §2.2]. □

*Remark* 5.3. Proceeding as above and interchanging the rules in (5.4) by

$$U = \mathrm{reg}(\mathbf{S}) \setminus (\mathfrak{B}')_{\mathrm{red}}, \quad V = \pi'^{-1}(U), \quad Z = U \cap (\mathfrak{B}')_{\mathrm{red}} = \emptyset, \tag{5.7}$$

we deduce that $\pi'|_V : V \to U$ is an isomorphism and in particular $\mathrm{Pic}(V) = \pi'^*(\mathrm{Pic}(U)) = \mathbb{Z}\langle H'\rangle$ and $K_V = \pi'^*(K_U) = (\Delta - n - 2)H'$. Now, by (5.5), $K_X|_V = K_V \sim (-2dr + r + dn - n - 3d + 1)H'$, and hence, also without to assume (5.1), we obtain that $r = (dn - \Delta - 3d + 3)\,/\,(2d-1)$.

Lemma 5.4 is slightly stronger than what is obtained by applying directly the main result in [BEL91]. However it is essential to study $\mathfrak{B}$ in the case $\delta = 0$. Note that the assumptions on $\mathbf{S}$ are not necessary.

**Lemma 5.4** ([MR05]). *For $i > 0$ and $t \geq n - 2r - 1$ we have $H^i(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B},\mathbb{P}^n}(t)) = 0$.*

*Proof.* The proof is located in [MR05, page 6], but we report it for the reader's convenience. We use the notation of the proof of Proposition 5.2. For $i > 0$ and $t \geq n-2r-1$ we have

$$\begin{array}{rcl}
H^i(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B},\mathbb{P}^n}(t)) & = & H^i(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(t)\otimes \mathscr{I}_{\mathfrak{B},\mathbb{P}^n}) \\
\text{([BEL91, page 592])} & = & H^i(X, tH-E) \\
& = & H^i(X, K_X + (tH-E-K_X)) \\
& = & H^i(X, K_X + (t+n+1)H + (r-n)E) \\
& = & H^i(X, K_X + (n-r)(2H-E) + (t-n+2r+1)H) \\
\text{([Deb01, page 20])} & = & H^i(X, K_X + \underbrace{\underbrace{(n-r)H'}_{\text{nef and big}} + \underbrace{(t-(n-2r-1))H}_{\text{nef}}}_{\text{nef and big}}) \\
\text{(Kodaira Vanishing Theorem)} & = & 0.
\end{array}$$

□

**Proposition 5.5.** *The following statements hold:*

1. *$\mathfrak{B}$ is nondegenerate and projectively normal.*
2. *If $\delta > 0$, then either $\mathfrak{B}$ is a Fano variety of the first species of index $i(\mathfrak{B}) = (r+\delta)/2$, or it is a hyperplane section of the Segre variety $\mathbb{P}^2\times\mathbb{P}^2 \subset \mathbb{P}^8$.*
3. *If $\delta > 0$, putting $i = i(\mathfrak{B})$ and $P(t) = P_{\mathfrak{B}}(t)$, we have*

$$\begin{gathered}
P(0)=1, \quad P(1)=n+1, \quad P(2)=\left(n^2+n-2\right)/2, \\
P(-1)=P(-2)=\cdots=P(-i+1)=0, \\
\forall t \quad P(t)=(-1)^r P(-t-i).
\end{gathered}$$

   *In particular, if $c(\mathfrak{B}) = r+1-i \leq 5$, it remains determined $P(t)$.*

4. *Hypothesis as in part 3. We have*

$$h^j(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(t)) = \begin{cases} 0 & \text{if} \quad j=0, \quad t<0 \\ P(t) & \text{if} \quad j=0, \quad t\geq 0 \\ 0 & \text{if} \quad 0<j<r, \quad t\in\mathbb{Z} \\ (-1)^r P(t) & \text{if} \quad j=r, \quad t<0 \\ 0 & \text{if} \quad j=r, \quad t\geq 0 \end{cases}$$

*Proof.* (1) $\mathfrak{B}$ is nondegenerate by Proposition 4.3. $\mathfrak{B}$ is projectively normal if and only if $h^1(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B},\mathbb{P}^n}(k)) = 0$ for every $k \geq 1$, and this, by Lemma 5.4, is true whenever $1 \geq n-2r-1$, i.e. whenever $\delta \geq 0$. Note that for $\delta > 0$ the thesis follows also from [BEL91, Corollary 2].

(2) We know that $\mathfrak{B}$ is a $QEL$-variety of type $\delta > 0$. If $\delta \geq 3$ the thesis is contained in Theorem 2.13; for $0 < \delta \leq 2$ we apply Theorem 2.19. Thus we have either that $\mathfrak{B}$ is a Fano variety with $\mathrm{Pic}(\mathfrak{B}) \simeq \mathbb{Z}\langle \mathscr{O}_{\mathfrak{B}}(1)\rangle$, or $\mathfrak{B}$ is one of the following:

1. $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$,

2. a rational normal scroll $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)^{\oplus r-1} \oplus \mathscr{O}(2)) \subset \mathbb{P}^{2r}$,

3. a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$,

4. $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$,

5. $\mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}$.

This follows only by imposing that the pair $(r,n)$ corresponds to that of a variety listed in Theorem 2.19. Of course cases 1, 4 and 5 are impossible, because $h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) = n+2$ but $h^0(\mathbb{P}^5, \mathscr{I}_{v_2(\mathbb{P}^2)}(2)) = 6$, $h^0(\mathbb{P}^8, \mathscr{I}_{\mathbb{P}^2\times\mathbb{P}^2}(2)) = 9$ and $h^0(\mathbb{P}^{11}, \mathscr{I}_{\mathbb{P}^2\times\mathbb{P}^3}(2)) = 18$. Case 2 is excluded because such a scroll satisfy[2] $\delta = 1$ (see for example [Rus10, Proposition 1.5.3]) and then we should have $n = 2r+1 \neq 2r$.

(3) By Lemma 5.4 we have $h^j(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B},\mathbb{P}^n}(k)) = 0$, for every $j > 0$, $k \geq 1-\delta$, and by the structural sequence we get $h^j(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(k)) = 0$, for every $j > 0$, $k \geq 1-\delta$. Hence, by $\delta > 0$, it follows

$$\begin{aligned} P(0) &= h^0(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) = 1, \\ P(1) &= h^0(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(1)) = h^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(1)) = n+1, \\ P(2) &= h^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(2)) - h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B},\mathbb{P}^n}(2)) = \left(n^2+n-2\right)/2. \end{aligned}$$

Moreover, if $t < 0$, by Kodaira Vanishing Theorem and Serre Duality,

$$P(t) = (-1)^r h^r(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(t)) = (-1)^r h^0(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(-t-i)),$$

and hence for $-i < t < 0$ we have $P(t) = 0$ and for $t \leq -i$ (hence for every $t$) we have $P(t) = (-1)^r P(-t-i)$. In particular

$$P(-i) = (-1)^r, \quad P(-i-1) = (-1)^r(n+1), \quad P(-i-2) = (-1)^r \left(n^2+n-2\right)/2,$$

and we have at least $i+5$ independent conditions for $P(t)$.

(4) As the part (3). □

## 5.2 Numerical restrictions

**Proposition 5.6.** *If $\Delta = 2$, then exactly one of the following holds:*

*(i)* *$d$ is even and $\delta \in \{0,1,3,7\}$;*

*(ii)* *$\delta$ is even, $d$ is odd and $d = u2^{\delta/2-1}+1$, for some $u \geq 1$;*

*(iii)* *$\delta = s2^e - 1$, $d = t2^{s2^{e-1}-e-1}+1$, for some $e,s,t \geq 1$, $s$ odd.*

[2] Actually one can easily see that if the base locus $\mathfrak{B} \subset \mathbb{P}^n$ is a rational normal scroll, then it holds $\delta = 0$, $r = 1$, $n = 4$, $d = 2$ and $\Delta = 2$.

*Proof.* By Theorem 2.14, for $\delta \geq 3$, we have $r \equiv \delta \bmod 2^{\lfloor(\delta-1)/2\rfloor}$, i.e.

$$\Delta - 2 + (d-1)(\delta+1) \equiv 0 \bmod 2^{\lfloor(\delta-1)/2\rfloor}. \tag{5.8}$$

If $d$ is even, then (5.8) becomes $\delta + 1 \equiv 0 \bmod 2^{\lfloor(\delta-1)/2\rfloor}$ and we conclude that $\delta \in \{0,1,2,3,7\}$. Moreover, if $\delta \geq 2$, by Proposition 5.5 we get that $\mathfrak{B}$ is a Fano variety with

$$2i(\mathfrak{B}) = r + \delta = \Delta - 4 + (d+1)(\delta+1) \equiv 0 \bmod 2, \tag{5.9}$$

and hence, if $\delta = 2$, we would get a contradiction. If $d$ is odd and $\delta$ is even, by (5.8) it immediately follows (ii). Finally, if $d$ and $\delta$ are both odd, we write $d = q2^a + 1$ and $\delta = s2^e - 1$, with $a, e, q, s \geq 1$ and $q, s$ odd. Then (5.8) is equivalent to $2^{a+e} \equiv 0 \bmod 2^{s2^{e-1}-1}$, i.e. $a \geq s2^{e-1} - e - 1$. Thus, putting $t = q2^{a-(s2^{e-1}-e-1)}$, we have $d = t2^{s2^{e-1}-e-1} + 1$. □

**Corollary 5.7.** *If $\Delta = 2$ and $d$ is even, then the possible values of $n$, $r$, $r'$ and $\delta$ are contained in Table 5.1.*

| $n$ | $r$ | $r'$ | $\delta$ |
|---|---|---|---|
| $2d$ | $d-1$ | $2(d-1)$ | $0$ |
| $4d-1$ | $2d-1$ | $4(d-1)$ | $1$ |
| $8d-3$ | $4d-1$ | $8(d-1)$ | $3$ |
| $16d-7$ | $8d-1$ | $16(d-1)$ | $7$ |

Table 5.1: Values of $n$, $r$, $r'$ and $\delta$, when $\Delta = 2$ and $d$ is even.

**Proposition 5.8.** *If $d$, $\Delta$ are both odd, then $\delta = 0$, $r = \Delta + d - 3$, $n = 2(\Delta + d - 2)$.*

*Proof.* If $\delta > 0$, by Proposition 5.5 we would get the contradiction that (5.9) holds. Thus $\delta = 0$ and there follow the expressions of $r$ and $n$ by Proposition 5.2. □

| $\Delta$ | $d$ | $(n, r, \delta, c)$ |
|---|---|---|
| 2 | 2 | $(4,1,0,\_)$, $(7,3,1,\_)$, $(13,7,3,3)$, $(25,15,7,5)$ |
|  | 3 | $(6,2,0,\_)$, $(16,8,2,4)$, $(21,11,3,5)$, $(26,14,4,6)$, $(31,17,5,7)$, $(41,23,7,9)$ |
|  | 4 | $(8,3,0,\_)$, $(15,7,1,4)$, $(29,15,3,7)$, $(57,31,7,13)$ |
| 3 | 2 | $(18,10,4,4)$, $(24,14,6,5)$ |
|  | 3 | $(8,3,0,\_)$ |
|  | 4 | $(10,4,0,\_)$, $(24,12,2,6)$, $(38,20,4,9)$ |
| 4 | 2 | $(17,9,3,4)$, $(23,13,5,5)$ |
|  | 3 | $(10,4,0,\_)$, $(15,7,1,4)$, $(20,10,2,5)$, $(25,13,3,6)$, $(30,16,4,7)$, $(40,22,6,9)$ |
|  | 4 | $(12,5,0,\_)$, $(19,9,1,5)$, $(33,17,3,8)$, $(47,25,5,11)$, $(75,41,9,17)$ |

Table 5.2: All cases with $\Delta \leq 4$ and $d \leq 4$.

*Remark* 5.9. Let us say a few words about the construction of Table 5.2. If $\Delta = 2$ and either $d = 2$ or $d = 4$, then the list of cases is obtained by Corollary 5.7. If $\Delta = 2$ and $d = 3$, then by (5.8) we obtain $(n,r,\delta) \in \{(6,2,0),(11,5,1),(16,8,2),(21,11,3),(26,14,4),(31,17,5),(41,23,7)\}$; the case $(n,r,\delta) = (11,5,1)$ is impossible, since otherwise by Proposition 5.5 part 3, we would get incompatible conditions for $P_{\mathfrak{B}}(t)$; by the same Proposition, we also get that in the case $(n,r,\delta) = (16,8,2)$ (resp. $(n,r,\delta) = (21,11,3)$) we have $\lambda = 36$ (resp. $\lambda = 86$). For the cases with $d = 2$ and either $\Delta = 3$ or $\Delta = 4$, we refer to Propositions 5.30 and 5.31, below. The unique case with $\Delta = 3$ and $d = 3$ is obtained by Proposition 5.8. Finally, cases with either $(\Delta,d) = (3,4)$, $(\Delta,d) = (4,3)$ or $(\Delta,d) = (4,4)$ are obtained by (5.8) and (5.9).

### 5.2.1 Sequences of impossible values of $n$

If we fix $\Delta$, then for each $n$ there are only finitely many possible values of $d$. So we can ask: given $\Delta$, what are the values of $n$ such that no $d$ works? In other words, *What are the values of $n$ such that there are no special quadratic birational transformations $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{S} \subset \mathbb{P}^{n+1}$ as in Assumption 5.1?*

In particular, let $\Delta = 2$. Then, for $n \leq 60$ all (non-excluded) values of $d$ are contained in Table 5.3, which is constructed via Proposition 5.6 and where an asterisk denotes cases further excluded by Proposition 5.5 part 3. We then see that for $n \in \{3,5,9,17,33\}$ no $d$ is possible.

| $n$ | $d$ | $n$ | $d$ | $n$ | $d$ | $n$ | $d$ | $n$ | $d$ | $n$ | $d$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 11 | 3* | 21 | 3 | 31 | 3, 8 | 41 | 3 | 51 | 13 |
| 2 | | 12 | 6 | 22 | 11 | 32 | 16 | 42 | 21 | 52 | 9, 26 |
| 3 | | 13 | 2 | 23 | 6 | 33 | | 43 | 11 | 53 | 7 |
| 4 | 2 | 14 | 7 | 24 | 12 | 34 | 17 | 44 | 22 | 54 | 27 |
| 5 | | 15 | 4 | 25 | 2 | 35 | 9 | 45 | 6 | 55 | 5, 14 |
| 6 | 3 | 16 | 3, 8 | 26 | 3, 13 | 36 | 18 | 46 | 5, 23 | 56 | 28 |
| 7 | 2 | 17 | | 27 | 7 | 37 | 5 | 47 | 12 | 57 | 4 |
| 8 | 4 | 18 | 9 | 28 | 5, 14 | 38 | 19 | 48 | 24 | 58 | 29 |
| 9 | | 19 | 5 | 29 | 4 | 39 | 10 | 49 | | 59 | 15 |
| 10 | 5 | 20 | 10 | 30 | 15 | 40 | 7, 20 | 50 | 25 | 60 | 30 |

Table 5.3: Values of $d$, when $\Delta = 2$ and $n \leq 60$.

More generally, if we define the sequence $\xi_2(k) = 1 + 2^k$, then we have that $n \neq \xi_2(k)$ for any $k$. In fact, if $n = \xi_2(k)$ for some $k$, then $(d-1)2^k = (2d-1)(r-\delta)$ and hence the contradiction that $(2d-1)$ divides $(d-1)$.

Obviously, when $n > 33$, there are many values of $n$ that are not possible and that do not belong to the image of $\xi_2$. So we define another sequence $\xi_2'(k)$, $k \geq 0$, as follows: $\xi_2'(0) = 33$, and, for $k \geq 1$,

$$\xi_2'(k) = \begin{cases} \xi_2'(k-1) + 16, & \text{if } k \not\equiv 0 \bmod 15, \\ \xi_2'(k-1) + 32, & \text{if } k \equiv 0 \bmod 15; \end{cases}$$

equivalently, for $k \geq 0$, we have $\xi_2'(k) = 33 + 16\lfloor k/15 \rfloor + 16k$.

**Proposition 5.10.** *Let $\Delta = 2$.*

1. *$n = \xi_2'(k)$, for some $k \geq 0$, if and only if case (iii) of Proposition 5.6 occurs, with either $e \geq 5$ or $e = 4$ and $s \not\equiv 1 \bmod 16$.*

2. *If $n = \xi_2'(k)$, then $k \geq 3840$ and $n \geq 65569$.*

3. *$n \neq \xi_2'(k)$, for infinitely many $k$.*

*Proof.* By straightforward calculation we see that $n = \xi_2'(k)$, for some $k \geq 0$, if and only if

$$a(n) := \frac{n-33}{16} \in \mathbb{Z} \quad \text{and} \quad b(n) := \frac{a(n)-15}{16} = \frac{n-273}{256} \notin \mathbb{Z}.$$

It follows that $\{\xi_2(k) : k \geq 5\} \subset \{\xi_2'(k) : k \geq 0\}$ and hence we have statement 3. Statement 2 follows from statement 1 and it can also be shown by computer. We now show statement 1. $\delta = 0,1,3,7$ respectively implies $n = d2^i - \delta$, with $i = 1,2,3,4$, and for these $n$ and $\delta$, $a(n) = \left(d2^i - \delta - 33\right)/16 = 2^{i-1}(2d-1)/16 - 2 \notin \mathbb{Z}$. If $\delta$ is even, then $n = (2d-1)(\delta+1)+1$ is even, but the values of $\xi_2'(k)$ are always odd. Thus, if $n = \xi_2'(k)$, then case (iii) of Proposition 5.6 occurs and we have

$$n = s\,2^{2^{e-1}s}t + 2^e s + 1, \qquad a(n) = 2^{e-4}s\left(2^{2^{e-1}s-e}t+1\right) - 2,$$
$$d = 2^{2^{e-1}s-e-1}t + 1, \qquad b(n) = s\,2^{2^{e-1}s-8}t + 2^{e-8}s - \frac{17}{16}.$$

If $e = 1$,

$$a(n) = \frac{s\left(2^{s-1}t+1\right)}{8} - 2 \in \mathbb{Z} \;\Rightarrow\; s = 1 \text{ and } t \text{ odd} \;\Rightarrow\; d = \frac{t}{2} + 1 \notin \mathbb{Z},$$

from which we obtain that $a(n) \notin \mathbb{Z}$. Similarly, we obtain that $a(n) \notin \mathbb{Z}$, if $e = 2$. If $e = 3$, since $2^{e-1}s - e \geq 1$, we again obtain that $a(n) \notin \mathbb{Z}$. While for $e \geq 4$, $a(n)$ is always integer. Finally, for $e = 4$,

$$b(n) = s\,2^{8s-8}t + \frac{s-1}{16} - 1 \notin \mathbb{Z} \;\Leftrightarrow\; \frac{s-1}{16} \notin \mathbb{Z},$$

and for $e \geq 5$, $b(n)$ is never integer. □

Now we consider the case in which $\Delta \neq 2$. Firstly, we observe that $n \geq 2\Delta$, since otherwise we should have $\delta = (n - 2\Delta - 2d + 4)/(2d-1) \leq -1 + 2/(2d-1) < 0$. Define the sequences $n = \xi_\Delta(k)$, $k \geq 0$, as follows:

$$\xi_\Delta(k) := 2\Delta - 3 + 2^{j(\Delta)} + 2^{j(\Delta)}k,$$

where

$$j(\Delta) = \begin{cases} 2 & \text{if } \Delta \equiv 1 \bmod 2, \\ \min\{j \geq 3 : \Delta \not\equiv 2 \bmod 2^j\} & \text{if } \Delta \equiv 0 \bmod 2. \end{cases}$$

Then we have the following:

**Proposition 5.11.** *If $\Delta \neq 2$, then $n \neq \xi_\Delta(k)$, for any $k \geq 0$.*

*Proof.* Suppose, by contradiction, that $n = \xi_\Delta(k)$, for some $k \geq 0$. We have $n = (2d-1)(\delta+1)+2\Delta-3$, so that $\delta$ must be odd. Now suppose that $\Delta \equiv 1 \bmod 2$. By Proposition 5.8 it follows that $d$ is even. If $\delta \geq 3$, by (5.8) we obtain the contradiction $0 \bmod 2 = (d-1)(\delta+1)+\Delta-2 \bmod 2 = 1 \bmod 2$, so we must have $\delta = 1$. But then $n = 2\Delta-5+4d = 2\Delta+1+4k$ and we obtain another contradiction, namely that $2(d-k) = 3$. Now suppose that $\Delta \equiv 0 \bmod 2$. By the fact that $\xi_\Delta(k) = n$, we deduce $(2d-1)(\delta+1) = 2^{j(\Delta)}(k+1)$, from which it follows that $\delta = -1+2^{j(\Delta)}h$, for some $h \geq 1$. So $\lfloor(\delta-1)/2\rfloor = (\delta-1)/2 = 2^{j(\Delta)-1}h-1 \geq 2^{j(\Delta)-1}-1 \geq j(\Delta)$ and by (5.8) we obtain $0 \bmod 2^{j(\Delta)} = (d-1)(\delta+1)+\Delta-2 \bmod 2^{j(\Delta)} = (d-1)2^{j(\Delta)}h+\Delta-2 \bmod 2^{j(\Delta)} = \Delta-2 \bmod 2^{j(\Delta)}$, against the definition of $j(\Delta)$. □

*Remark* 5.12. Although our original hypothesis is $\Delta \geq 2$, note that also when $\Delta = 1$, Proposition 5.11 is still valid. In other words, *There are no special quadratic Cremona transformations of $\mathbb{P}^n$, whenever $n \in \{3,7,11,15,19,\ldots\}$*. We should also note that, when $j(\Delta) \geq 5$, then Proposition 5.11 is very far from being sharp.

| $\Delta$ | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 9 | 17 | 33 | 49 | 65 | 81 | 97 | 113 | 129 | 145 | 161 | 177 | 193 | 209 | 225 | 241 | 257 | 289 |
| 3 | 3 | 4 | 5 | 7 | 11 | 15 | 19 | 23 | 27 | 31 | 35 | 39 | 43 | 47 | 51 | 55 | 59 | 63 | 67 | 71 |
| 4 | 3 | 4 | 5 | 6 | 7 | 9 | 13 | 21 | 29 | 37 | 45 | 53 | 61 | 69 | 77 | 85 | 93 | 101 | 109 | 117 |
| 5 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 11 | 15 | 19 | 23 | 27 | 31 | 35 | 39 | 43 | 47 | 51 | 55 | 59 |
| 6 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 13 | 17 | 25 | 33 | 41 | 49 | 57 | 65 | 73 | 81 | 89 |
| 7 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 15 | 19 | 23 | 27 | 31 | 35 | 39 | 43 | 47 |
| 8 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 17 | 21 | 29 | 37 | 45 | 53 | 61 |
| 9 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 19 | 23 | 27 | 31 | 35 |
| 10 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 21 | 25 | 33 |
| 11 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 23 |

Table 5.4: Some impossible values of $n$, when $2 \leq \Delta \leq 11$.

## 5.3 Examples

The calculations in the following examples can be verified with the aid of the computer algebra systems: [GS10] and [So11].

**Example 5.13** ($\Delta = 2, d = 2$)**.** Let $\psi : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n+1}$ be a Cremona transformation of type $(2,2)$. If $H \simeq \mathbb{P}^n \subset \mathbb{P}^{n+1}$ is a general hyperplane, then $\mathbf{Q} := \overline{\psi(H)} \subset \mathbb{P}^{n+1}$ is a quadric hypersurface and the restriction $\psi|_H : \mathbb{P}^n \dashrightarrow \mathbf{Q} \subset \mathbb{P}^{n+1}$ is a birational transformation of type $(2,2)$, with base locus $\mathrm{Bs}(\psi|_H) = \mathrm{Bs}(\psi) \cap H$. If $\psi$ is special, i.e. if its base locus is a Severi variety, then also $\psi|_H$ is special and moreover it is possible to verify that the quadric $\mathbf{Q}$ is smooth, for example by determining explicitly the equation.

**Example 5.14** ($\Delta = 2$, $d = 3$, $\delta = 0$)**.** We construct the Edge variety $\mathscr{X}$ of dimension 3 and degree 7 (see also [Edg32] and [CMR04, Example 2.4]). Consider the Segre variety $S^{1,3} = \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ and choose coordinates $x_0, \ldots, x_7$ on $\mathbb{P}^7$ such that the equations of $S^{1,3}$ in $\mathbb{P}^7$ are given by

$$\begin{array}{lll} Q_0 = -x_1x_4 + x_0x_5, & Q_1 = -x_2x_4 + x_0x_6, & Q_2 = -x_3x_4 + x_0x_7, \\ Q_3 = -x_2x_5 + x_1x_6, & Q_4 = -x_3x_5 + x_1x_7, & Q_5 = -x_3x_6 + x_2x_7. \end{array}$$

Take a general quadric $V(Q)$ containing the linear space $P = V(x_0, x_1, x_2, x_3) \subset S^{1,3}$, i.e. $Q = \sum b_{ij}x_ix_j$, for suitable coefficients $b_{ij} \in \mathbb{C}$, with $i \leq j$, $0 \leq i \leq 3$ and $0 \leq j \leq 7$. The intersection $\mathscr{Y} = S^{1,3} \cap V(Q)$ is an equidimensional variety of dimension 3 that has $P$ as irreducible component of multiplicity 1. Hence it defines a variety $\mathscr{X}$ of dimension 3 and degree 7 such that $\mathscr{Y} = P \cup \mathscr{X}$ and $P \nsubseteq \mathscr{X}$. Since we are interested in $\mathscr{Y}$ and not in $Q$, we can suppose $b_{14} = b_{24} = b_{34} = b_{25} = b_{35} = b_{36} = 0$. Thus it is easy to verify that $\mathscr{X}$ is the scheme-theoretic intersection of $\mathscr{Y}$ with the quadric $V(Q')$, where

$$\begin{array}{rcl} Q' & = & b_{37}x_7^2 + b_{27}x_6x_7 + b_{17}x_5x_7 + b_{07}x_4x_7 + b_{33}x_3x_7 + b_{26}x_6^2 + b_{16}x_5x_6 + \\ & + & b_{06}x_4x_6 + b_{23}x_3x_6 + b_{22}x_2x_6 + b_{15}x_5^2 + b_{05}x_4x_5 + b_{13}x_3x_5 + b_{12}x_2x_5 + \\ & + & b_{11}x_1x_5 + b_{04}x_4^2 + b_{03}x_3x_4 + b_{02}x_2x_4 + b_{01}x_1x_4 + b_{00}x_0x_4. \end{array}$$

The rational map $\psi : \mathbb{P}^7 \dashrightarrow \mathbb{P}^7$, defined by $\psi([x_0, \ldots, x_7]) = [Q_0, \ldots, Q_5, Q, Q']$, has as image the rank 6 quadric $\mathbf{Q} = V(y_0y_5 - y_1y_4 + y_2y_3)$ and the closure of its general fiber is a $\mathbb{P}^1 \subset \mathbb{P}^7$. Hence restricting $\psi$ to a general $\mathbb{P}^6 \subset \mathbb{P}^7$, we get a birational transformation $\mathbb{P}^6 \dashrightarrow \mathbf{Q} \subset \mathbb{P}^7$.

**Example 5.15.** Continuing with the Example 5.14, we set $b_{00} = b_{07} = b_{15} = b_{16} = b_{22} = b_{33} = 1$ and for all the other indices $b_{ij} = 0$. Substitute in the quadrics $Q_0, \ldots, Q_5, Q, Q'$ instead of the variable $x_7$ the variable $x_0$, i.e. we consider the intersection of $\mathscr{X}$ with the hyperplane $V(x_7 - x_0)$. Denote by $X \subset \mathbb{P}^6$ the scheme so obtained. $X$ is irreducible, smooth, it is defined by the 8 independent quadrics:

$$\begin{gathered} x_3x_6 - x_0x_2,\ x_5x_6 + x_2x_6 + x_5^2 + 2x_0x_4 + x_0x_3,\ x_3x_5 - x_0x_1,\ x_2x_5 - x_1x_6, \\ x_3x_4 - x_0^2,\ x_2x_4 - x_0x_6,\ x_1x_4 - x_0x_5,\ x_1x_6 + x_1x_5 + x_3^2 + x_2^2 + 2x_0^2, \end{gathered} \tag{5.10}$$

and its Hilbert polynomial is $P_X(t) = (7t^2 + 5t + 2)/2$. The quadrics (5.10) define a birational map $\psi : \mathbb{P}^6 \dashrightarrow \mathbf{Q} \subset \mathbb{P}^7$ into the quadric $\mathbf{Q} = V(y_0y_6 - y_2y_5 + y_3y_4)$ and the inverse of $\psi$ is defined by the cubics:

$$\begin{gathered} y_0y_5y_7 - y_1y_4y_7 + y_1y_2y_6 + y_0y_1y_6 + y_2y_3y_5 + y_0y_3y_5 + y_2^2y_4 + y_0y_2y_4, \\ -y_6y_7^2 - 2y_4y_6y_7 - y_3y_6y_7 + y_0y_3y_7 - y_1y_2y_7 + 2y_2y_6^2 + 2y_0y_6^2 + y_2y_3^2 + y_0y_3^2 + y_2^3 + y_0y_2^2, \\ -y_5y_7^2 - 2y_4y_5y_7 - y_3y_5y_7 - y_0y_1y_7 + 2y_2y_5y_6 + 2y_0y_5y_6 - y_1y_2y_3 - y_0y_1y_3 + y_0y_2^2 + y_0^2y_2, \\ -y_4y_7^2 + y_0y_6y_7 - y_2y_5y_7 - 2y_4^2y_7 - y_0^2y_7 + 2y_2y_4y_6 + 2y_0y_4y_6 - y_0y_2y_3 - y_0^2y_3 - y_1y_2^2 - y_0y_1y_2, \\ -y_5^2y_7 - y_4^2y_7 - y_1y_6^2 - y_3y_5y_6 - y_1y_5y_6 - y_2y_4y_6 - 2y_4y_5^2 - y_3y_5^2 - y_2y_4y_5 - 2y_4^3 - y_1^2y_4 - y_0^2y_4, \\ -y_1y_6y_7 - y_3y_5y_7 - y_2y_4y_7 - y_1y_3y_6 + y_0y_2y_6 - 2y_2y_5^2 - y_3^2y_5 - y_2^2y_5 - 2y_2y_4^2 - y_1^2y_2 - y_0^2y_2, \\ -y_1y_5y_7 - y_0y_4y_7 + y_1y_3y_6 - y_0y_2y_6 - 2y_0y_5^2 + y_3^2y_5 + y_2^2y_5 - 2y_0y_4^2 - y_0y_1^2 - y_0^3. \end{gathered} \tag{5.11}$$

The base locus $Y \subset \mathbf{Q} \subset \mathbb{P}^7$ of $\psi^{-1}$ is obtained by intersecting the scheme defined by (5.11) with the quadric $\mathbf{Q}$. $Y$ is irreducible with Hilbert polynomial

$$P_Y(t) = \frac{9t^4 + 38t^3 + 63t^2 + 58t + 24}{4!}$$

and its singular locus has dimension 0.

**Example 5.16** ($\Delta = 2$, $d = 4$, $\delta = 0$)**.** The 10 quadrics (2.10) define a rational map $\psi : \mathbb{P}^{15} \dashrightarrow \mathbb{P}^9$, with base locus $S^{10}$ and image in $\mathbb{P}^9$ the smooth quadric

$$\mathbf{Q} = V(y_8y_9 - y_6y_7 - y_0y_5 + y_2y_4 + y_1y_3).$$

By [ESB89, page 798] the closure of the general fiber of $\psi$ is a $\mathbb{P}^7 \subset \mathbb{P}^{15}$ and hence, by restricting $\psi$ to a general $\mathbb{P}^8 \subset \mathbb{P}^{15}$, we get a special birational transformation $\mathbb{P}^8 \dashrightarrow \mathbf{Q}$ (necessarily of type $(2,4)$, by Remark 5.3).

**Example 5.17** ($\Delta = 3$, $d = 3$)**.** Let $u : \mathbb{P}^3 \overset{\simeq}{\dashrightarrow} Q = V(u_0u_4 - u_1^2 - u_2^2 - u_3^2) \subset \mathbb{P}^4$ be defined by $u([z_0,z_1,z_2,z_3]) = [z_0^2, z_0z_1, z_0z_2, z_0z_3, z_1^2+z_2^2+z_3^2]$. Consider the composition $\nu^0 : \mathbb{P}^3 \overset{u}{\dashrightarrow} \mathbb{P}^4 \xrightarrow{\nu_2} \mathbb{P}^{14} \dashrightarrow \mathbb{P}^{13}$, where $\nu_2$ is the Veronese map (with lexicographic order) and the last map is the projection onto the hyperplane $V(v_4) \simeq \mathbb{P}^{13} \subset \mathbb{P}^{14}$. The map $\nu^0$ parameterizes a nondegenerate variety in $\mathbb{P}^{13}$, of degree 16 and isomorphic to the quadric $Q$. Take the point $p_1 = [1,0,0,0] \in \mathbb{P}^3$ and consider the composition $\nu^1 : \mathbb{P}^3 \overset{\nu^0}{\dashrightarrow} \mathbb{P}^{13} \overset{\pi_1}{\dashrightarrow} \mathbb{P}^{12}$, where $\pi_1$ is the projection from the point $\nu^0(p_1)$ (precisely, if $j$ is the index of the last nonzero coordinate of $\nu^0(p_1)$, we exchange the coordinates $v_j$, $v_{13}$ and we project onto the hyperplane $V(v_{13})$ from the point $\nu^0(p_1)$). Repeat the construction with the points $p_2 = [0,0,0,1]$, $p_3 = [1,0,0,1]$, $p_4 = [0,1,0,1]$, $p_5 = [0,0,1,1]$, obtaining the maps $\nu^2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{11}$, $\nu^3 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{10}$, $\nu^4 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^9$, $\nu^5 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^8$. The map $\nu^5$ is given by[3]

$$\begin{aligned} \nu^5([z_0,z_1,z_2,z_3]) \;=\; & [z_3^3 - z_0z_3^2 + z_2^2z_3 + z_1^2z_3,\ -z_1z_3^2 - z_1z_2^2 - z_1^3 + z_0^2z_1, \\ & -z_2z_3^2 - z_2^3 - z_1^2z_2 + z_0^2z_2,\ z_0^2z_3 - z_0z_3^2,\ -z_1z_3^2 - z_1z_2^2 - z_1^3 + z_0z_1^2, \\ & z_0z_1z_2,\ z_0z_1z_3,\ z_0z_2z_3,\ -z_2z_3^2 - z_2^3 + z_0z_2^2 - z_1^2z_2] \end{aligned}$$

and parameterizes the smooth variety $X \subset \mathbb{P}^8$ defined by the 10 independent quadrics:

$$\begin{gathered} x_5x_8 - x_4x_8 + x_1x_8 - x_4x_7 + x_5x_6 + x_2x_6 + x_5^2 + x_4x_5 - x_3x_5 - x_2x_5, \\ x_5x_8 - x_4x_8 - x_4x_7 + x_5x_6 + x_2x_6 + x_5^2 + x_4x_5 - x_3x_5 - x_1x_5 + x_2x_4, \\ -x_7x_8 - x_0x_8 - x_7^2 + x_3x_7 - x_5x_6 + x_0x_2, \\ x_6x_8 - x_4x_7 + x_6^2 + x_5x_6 + x_4x_6 - x_1x_6 - x_0x_6 + x_3x_4, \\ x_6x_7 + x_4x_7 - x_5x_6 - x_2x_6 + x_3x_5, \\ x_7x_8 + x_3x_8 + x_7^2 - x_2x_7 - x_0x_7 + x_5x_6, \\ x_1x_7 - x_2x_6, \\ x_5x_7 + x_6^2 + x_4x_6 - x_1x_6 - x_0x_6 + x_3x_4, \\ x_2x_6 - x_3x_5 + x_0x_5, \\ x_3x_6 - x_1x_6 - x_0x_6 + x_3x_4 - x_0x_4 + x_0x_1. \end{gathered} \tag{5.12}$$

[3]Note that $\nu^5 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^8$ can be obtained by choosing a suitable basis of $H^0(\mathbb{P}^3, \mathscr{I}_Z(3))$, where $Z$ is the union of the conic $V(z_0, z_1^2+z_2^2+z_3^2)$ and the four points: $[1,0,0,0]$, $[1,1,0,0]$, $[1,0,1,0]$, $[1,0,0,1]$.

The quadrics (5.12) define a special birational transformation $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^9$ of type $(2,3)$ into the cubic $\mathbf{S} \subset \mathbb{P}^9$ defined by

$$\begin{aligned}
&-y_7 y_8 y_9 + y_6 y_8 y_9 + y_3 y_8 y_9 + y_7^2 y_9 - y_6 y_7 y_9 - 2 y_3 y_7 y_9 - y_2 y_7 y_9 + \\
&+ y_0 y_7 y_9 + y_4 y_6 y_9 + y_3 y_6 y_9 - y_4 y_5 y_9 - y_1 y_5 y_9 - y_4^2 y_9 - y_0 y_4 y_9 + \\
&+ y_3^2 y_9 + y_2 y_3 y_9 - y_0 y_3 y_9 + y_7 y_8^2 - y_3 y_8^2 - y_6 y_7 y_8 + 2 y_4 y_7 y_8 + \\
&+ y_3 y_7 y_8 - y_2 y_7 y_8 - y_4 y_6 y_8 + y_2 y_6 y_8 - y_0 y_6 y_8 + y_4 y_5 y_8 + y_0 y_5 y_8 + \\
&-3 y_3 y_4 y_8 - y_1 y_4 y_8 + y_0 y_4 y_8 - y_3^2 y_8 + y_2 y_3 y_8 - y_1 y_3 y_8 + y_5 y_7^2 + \\
&+ y_2 y_7^2 + y_1 y_7^2 - y_0 y_7^2 + y_6^2 y_7 + y_3 y_6 y_7 + y_0 y_6 y_7 - y_3 y_5 y_7 + \\
&+ y_1 y_5 y_7 - y_2 y_3 y_7 - y_1 y_3 y_7 + y_0 y_3 y_7 + y_0 y_2 y_7 - y_4 y_6^2 - y_3 y_6^2 + \\
&- y_3 y_5 y_6 - y_3 y_4 y_6 - y_0 y_4 y_6 - y_3^2 y_6 - y_0 y_3 y_6 - y_1 y_2 y_6 + y_4^2 y_5 + \\
&+ y_3 y_4 y_5 + y_0 y_4 y_5 + y_2 y_4^2 - y_1 y_4^2 + y_0 y_4^2 + y_2 y_3 y_4 - y_1 y_3 y_4 + \\
&+ y_0 y_3 y_4 + y_1 y_2 y_4 .
\end{aligned} \tag{5.13}$$

The singular locus of $\mathbf{S}$ has dimension 3, from which it follows the factoriality of $\mathbf{S}$ (see [Gro68, XI Corollaire 3.14] and Remark 6.7 below).

**Example 5.18** ($\Delta = 4$, $d = 2$, $\delta = 0$)**.** This is an example for which (5.1) is not satisfied; essentially, it gives an example for case (ii) of Proposition 5.23. Consider the irreducible smooth 3-fold $X \subset \mathbb{P}^8$ defined as the intersection of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, which is defined by (2.5), with the quadric hypersurface defined by:

$$2x_0^2 + 3x_1^2 + 5x_2^2 + x_3^2 + x_4^2 + x_5^2 + 5x_6^2 + 3x_7^2 + 2x_8^2. \tag{5.14}$$

Thus $X$ is defined by 10 quadrics and we can consider the associated rational map $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} = \overline{\psi(\mathbb{P}^8)} \subset \mathbb{P}^9$. We have that $\mathbf{S}$ is the quartic hypersurface defined by:

$$\begin{aligned}
&2y_1^2y_2^2 + 3y_1^2y_3^2 - 4y_0y_1y_2y_4 + 2y_0^2y_4^2 + 5y_3^2y_4^2 - 6y_0y_1y_3y_5 - 10y_2y_3y_4y_5 + 3y_0^2y_5^2 \\
&+5y_2^2y_5^2 + y_2^2y_6^2 + y_3^2y_6^2 + 5y_4^2y_6^2 + 3y_5^2y_6^2 - 2y_0y_2y_6y_7 - 10y_1y_4y_6y_7 + y_0^2y_7^2 \\
&+5y_1^2y_7^2 + y_3^2y_7^2 + 2y_5^2y_7^2 - 2y_0y_3y_6y_8 - 6y_1y_5y_6y_8 - 2y_2y_3y_7y_8 - 4y_4y_5y_7y_8 + y_0^2y_8^2 \\
&+3y_1^2y_8^2 + y_2^2y_8^2 + 2y_4^2y_8^2 - y_3y_4y_6y_9 + y_2y_5y_6y_9 + y_1y_3y_7y_9 - y_0y_5y_7y_9 - y_1y_2y_8y_9 + y_0y_4y_8y_9,
\end{aligned} \tag{5.15}$$

and $\psi$ is birational with inverse defined by:

$$\begin{gathered}
y_5y_7 - y_4y_8,\ y_5y_6 - y_1y_8,\ y_4y_6 - y_1y_7,\ y_3y_7 - y_2y_8,\ y_3y_6 - y_0y_8, \\
y_2y_6 - y_0y_7,\ y_3y_4 - y_2y_5,\ y_1y_3 - y_0y_5,\ y_1y_2 - y_0y_4.
\end{gathered} \tag{5.16}$$

Moreover $X$ is a Mukai variety of degree 12, sectional genus 7 and Betti numbers $b_2 = 2$, $b_3 = 18$. The secant variety $\mathrm{Sec}(X)$ is a cubic hypersurface and for the general point $p \in \mathrm{Sec}(X)$ there are two secant lines of $X$ through $p$. Finally, denoting by $Y \subset \mathbf{S}$ the base locus of $\psi^{-1}$, we have $\dim(Y) = 5$, $\dim(\mathrm{sing}(Y)) = 0$ and $Y = (Y)_{\mathrm{red}} = (\mathrm{sing}(\mathbf{S}))_{\mathrm{red}}$.

## 5.4 Transformations of type $(2,2)$ into a quadric

**Theorem 5.19.** *If $\Delta = 2$, $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{Q} := \mathbf{S} \subset \mathbb{P}^{n+1}$ is of type $(2,2)$ and $\mathbf{Q}$ is smooth, then $\mathfrak{B}$ is a hyperplane section of a Severi variety.*

*Proof.* By Corollary 5.7 we get $\delta \in \{0,1,3,7\}$. If $\delta = 0$, we have $r = 1$, $n = 4$ and the thesis follows from Proposition 4.8. Alternatively, we can apply Lemma 5.4 to determine the Hilbert polynomial of the curve $\mathfrak{B}$. If $\delta = 1$, we have $r = 3$, $n = 7$ and, by Proposition 5.5 part 2, $\mathfrak{B}$ is a hyperplane section of the Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, or it is a Fano variety of the first species of index $i(\mathfrak{B}) = r - 1$. The latter case cannot occur by the classification of del Pezzo varieties in Theorem 2.7. If $\delta = 3$, we have $r = 7$, $n = 13$ and $\mathfrak{B}$ is a Mukai variety of the first specie. By Proposition 5.5 part 3, we can determine the Hilbert polynomial of $\mathfrak{B}$ and in particular to get that the sectional genus is $g = 8$. Hence, applying the classification of Mukai varieties in Theorem 2.8, we get that $\mathfrak{B}$ is a hyperplane section of the Grassmannian $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$. Now, suppose $\delta = 7$ and hence $r = 15$, $n = 25$, $r' = 16$. Keep the notation as in the proof of Proposition 5.2 and put $Y = (\mathfrak{B}')_{\mathrm{red}}$. We shall show that $\mathfrak{B} \subset \mathbb{P}^{25}$ is a hyperplane section of $E_6 \subset \mathbb{P}^{26}$ in several steps.

*Claim* 5.19.1. $\pi'(E) = (\mathrm{Sec}(\mathfrak{B}') \cap \mathbf{Q})_{\mathrm{red}} = (\mathrm{Sec}(Y) \cap \mathbf{Q})_{\mathrm{red}}$ and for the general point $p \in \pi'(E)$ the entry locus $\Sigma_p(\mathfrak{B}')$ is a quadric of dimension 8.

*Proof of the Claim.* See also [ESB89, Proposition 2.3]. If $x \in \mathfrak{B}$ is a point, then $\pi'(\pi^{-1}(x)) \simeq \mathbb{P}^9$ and by the relations (5.3) we get that $\pi'(\pi^{-1}(x)) \cap \mathfrak{B}'$ is a quadric hypersurface in $\mathbb{P}^9$. Then, for every point $p \in \pi'(\pi^{-1}(x)) \setminus \mathfrak{B}'$, every line contained in $\pi'(\pi^{-1}(x))$ and through $p$ is a secant line of $\mathfrak{B}'$ and therefore it is contained in $L_p(\mathfrak{B}')$. Thus

$$\pi'(\pi^{-1}(x)) \subseteq L_p(\mathfrak{B}') \subseteq \mathrm{Sec}(\mathfrak{B}') \cap \mathbf{Q}$$

and varying $x \in \mathfrak{B}$ we get

$$\pi'(E) = \bigcup_{x \in \mathfrak{B}} \pi'(\pi^{-1}(x)) \subseteq \mathrm{Sec}(\mathfrak{B}') \cap \mathbf{Q}. \tag{5.17}$$

Conversely, if $p \in \mathrm{Sec}(\mathfrak{B}') \cap \mathbf{Q} \setminus \mathfrak{B}'$, then every secant line $l$ of $\mathfrak{B}'$ through $p$ is contained in $\mathbf{Q}$ and $\varphi^{-1}(l) = \varphi^{-1}(p) =: x$. Hence the strict transform $\widetilde{l} = \overline{\pi'^{-1}(l \setminus \mathfrak{B}')}$ is contained in $\pi^{-1}(x) \subseteq E$ and then $l$ is contained in $\pi'(\pi^{-1}(x)) \subseteq \pi'(E)$. Thus

$$L_p(\mathfrak{B}') \subseteq \pi'(\pi^{-1}(x)) \subseteq \pi'(E)$$

and varying $p$ (since $p$ lies on at least a secant line of $\mathfrak{B}'$) we get

$$\left(\mathrm{Sec}(\mathfrak{B}') \cap \mathbf{Q}\right)_{\mathrm{red}} = \overline{(\mathrm{Sec}(\mathfrak{B}') \cap \mathbf{Q} \setminus \mathfrak{B}')_{\mathrm{red}}} \subseteq \pi'(E). \tag{5.18}$$

The conclusion follows from (5.17) and (5.18) and by observing that, since $\mathfrak{B}'$ is irreducible and generically reduced, we have $(\mathrm{Sec}(\mathfrak{B}'))_{\mathrm{red}} = \mathrm{Sec}(Y)$. $\square$

*Claim* 5.19.2. For the general point $x \in \mathfrak{B}$, $\overline{\tau_{x,\mathfrak{B}}(\mathfrak{B})} = W_{x,\mathfrak{B}} \simeq \Sigma_p(\mathfrak{B}')$, with $p$ general point in $\pi'(\pi^{-1}(x))$.

*Proof of the Claim.* See also [MR05, Theorem 1.4]. The projective space $\mathbb{P}^9 \subset \mathbb{P}^{25}$ skew to $T_x(\mathfrak{B})$ (i.e. the codomain of $\tau_{x,\mathfrak{B}}$) identifies with $\pi^{-1}(x) = \mathbb{P}(\mathscr{N}^*_{x,\mathfrak{B}})$. For $z \in \mathfrak{B} \setminus T_x(\mathfrak{B})$, the line $l = \langle z,x \rangle$ corresponds to a point $w \in \pi^{-1}(x) = \mathbb{P}(\mathscr{N}^*_{x,\mathfrak{B}})$. Moreover $\pi'(w) = \varphi(l) \in \mathfrak{B}'$

and $\tau_{x,\mathfrak{B}}(z) = \tau_{x,\mathfrak{B}}(l) = w$, so that $\pi'(w) \in \pi'(\pi^{-1}(x)) \cap \mathfrak{B}'$ and hence $W_{x,\mathfrak{B}}$ identifies with an irreducible component of $\pi'(\pi^{-1}(x)) \cap \mathfrak{B}'$. Since $W_{x,\mathfrak{B}}$ is a nondegenerate hypersurface in $\mathbb{P}(\mathscr{N}^*_{x,\mathfrak{B}})$ (see Proposition 1.14), we have

$$W_{x,\mathfrak{B}} \simeq \pi'(\pi^{-1}(x)) \cap \mathfrak{B}' \subset \pi'(\pi^{-1}(x)) \simeq \mathbb{P}^9. \tag{5.19}$$

□

*Claim* 5.19.3. For the general point $x \in \mathfrak{B}$, $W_{x,\mathfrak{B}} \subset \mathbb{P}^9$ is a smooth quadric hypersurface.

*Proof of the Claim.* By Theorem 2.13 it follows that $\mathscr{L}_{x,\mathfrak{B}} \subset \mathbb{P}^{14}$ is a smooth *QEL*-variety of dimension $\dim(\mathscr{L}_{x,\mathfrak{B}}) = 9$ and of type $\delta(\mathscr{L}_{x,\mathfrak{B}}) = 5$. So, applying Theorem 2.17, we get that $\mathscr{L}_{x,\mathfrak{B}} \subset \mathbb{P}^{14}$ is projectively equivalent to a hyperplane section of the spinorial variety $S^{10}$. Hence, by Proposition 1.16 and Example 5.16, it follows that $W_{x,\mathfrak{B}} \subset \mathbb{P}^9$ is a smooth quadric of dimension $r - \delta = 8$. □

Summing up the previous claims, we obtain that for the general point $p \in \pi'(E) \setminus Y$ the entry locus $\Sigma_p(\mathfrak{B}')$ is a smooth quadric; moreover, one can be easily convinced that we also have $\Sigma_p(\mathfrak{B}') = L_p(\mathfrak{B}') \cap \mathfrak{B}' = L_p(\mathfrak{B}') \cap Y = L_p(Y) \cap Y = \Sigma_p(Y)$. Now, by the proof of Proposition 5.2, it follows that $\pi'(E)$ is a reduced and irreducible divisor in $|\mathscr{O}_{\mathbf{Q}}(3)|$ and, since $\pi'(E) = (\mathrm{Sec}(Y) \cap \mathbf{Q})_{\mathrm{red}}$, we have that $\mathrm{Sec}(Y)$ is a hypersurface of degree a multiple of 3. In particular $Y \subset \mathbb{P}^{26}$ is an irreducible nondegenerate variety with secant defect $\delta(Y) = 8$. Note also that $Y$ is different from a cone: if $z_0$ is a vertex of $Y$, since $I_{\mathfrak{B}',\mathbb{P}^{26}} \subseteq I_{Y,\mathbb{P}^{26}}$, we have that $z_0$ is a vertex of every quadric defining $\mathfrak{B}'$, in particular $z_0$ is a vertex of $\mathfrak{B}'$ and $\mathbf{Q}$; then, for a general point $z \in \mathbf{Q}$, we have $\langle z, z_0 \rangle \subseteq \overline{(\varphi^{-1})^{-1}(\varphi^{-1}(z))}$, against the birationality of $\varphi^{-1}$.

*Claim* 5.19.4. $Y$ is a *QEL*-variety.

*Proof of the Claim.* Consider the rational map $\psi : \mathrm{Sec}(Y) \dashrightarrow \mathbb{P}^{26}$, defined by the linear system $|\mathscr{I}_{\mathfrak{B}',\mathbb{P}^{26}}(2)|$ restricted to $\mathrm{Sec}(Y)$. Since $\overline{\psi(\mathrm{Sec}(Y) \cap \mathbf{Q})} = \overline{\varphi^{-1}(\pi'(E))} = \mathfrak{B}$ and the image of $\psi$ is of course nondegenerate, it follows that the dimension of the image of $\psi$ is at least 16 and hence that the dimension of its general fiber is at most 9. Now, if $q \in \mathrm{Sec}(Y) \setminus Y$ is a general point, denote by $\widetilde{\psi^{-1}(\psi(q))}$ the irreducible component of $\overline{\psi^{-1}(\psi(q))}$ through $q$ and by $\widetilde{\Sigma_q(Y)}$ an any irreducible component of $\Sigma_q(Y)$; note that, from generic smoothness [Har77, III Corollary 10.7], it follows that $\overline{\psi^{-1}(\psi(q))}$ (and at fortiori $\widetilde{\psi^{-1}(\psi(q))}$) is smooth in its general point $q$. We have $S(q, \widetilde{\Sigma_q(Y)}) \subseteq \widetilde{\psi^{-1}(\psi(q))}$, and since $\dim(\widetilde{\Sigma_q(Y)}) = 8$, it follows $S(q, \widetilde{\Sigma_q(Y)}) = \widetilde{\psi^{-1}(\psi(q))}$. Thus the cone $S(q, \widetilde{\Sigma_q(Y)})$ is smooth in its vertex and necessarily it follows $S(q, \widetilde{\Sigma_q(Y)}) = \mathbb{P}^9$ and $\langle \Sigma_q(Y) \rangle = \mathbb{P}^9$. Finally, by Trisecant Lemma [Rus10, Proposition 1.3.3], it follows that $\Sigma_q(Y) \subset \mathbb{P}^9$ is a quadric hypersurface. □

*Claim* 5.19.5. $\widetilde{\gamma}(Y) = 0$.

*Proof of the Claim.* We first show that for the general point $q \in \mathrm{Sec}(Y) \setminus Y$ the entry locus $\Sigma_q(Y)$ is smooth, by discussing two cases:

*Case* 5.19.1 (Suppose $\pi'(E) \nsubseteq \mathrm{sing}(\mathrm{Sec}(Y))$). Denote by $\mathrm{Hilb}(Y)$ the Hilbert scheme of 8-dimensional quadrics contained in $Y$ and by $V$ a nonempty open set of $\mathrm{Sec}(Y) \setminus Y$ such that for every $q \in V$ we have $\Sigma_q(Y) \in \mathrm{Hilb}(Y)$. If $\rho : Y \times (Y \times \mathbb{P}^{26}) \longrightarrow Y \times \mathbb{P}^{26}$ is the projection, at the closed subscheme of $Y \times V$,

$$\rho\left(\overline{\{(w,z,q) \in Y \times Y \times \mathbb{P}^{26} : w \neq z \text{ and } q \in \langle w,z \rangle\}}\right) \cap Y \times V = \{(z,q) \in Y \times V : z \in \Sigma_q(Y)\},$$

corresponds a rational map $\nu : \mathrm{Sec}(Y) \dashrightarrow \mathrm{Hilb}(Y)$ which sends the point $q \in V$ to the quadric $\nu(q) = \Sigma_q(Y)$; denote by $\mathrm{Dom}(\nu)$ the largest open set of $\mathrm{Sec}(Y)$ where $\nu$ can be defined. By assumption we have $D' := \pi'(E) \cap \mathrm{reg}(\mathrm{Sec}(Y)) \neq \emptyset$. It follows that the rational map $\nu' := \nu|_{\mathrm{reg}(\mathrm{Sec}(Y))} : \mathrm{reg}(\mathrm{Sec}(Y)) \dashrightarrow \mathrm{Hilb}(Y)$, having indeterminacy locus of codimension $\geq 2$, is defined in the general point of $D'$, i.e.

$$\emptyset \neq \mathrm{Dom}(\nu') \cap D' = \mathrm{Dom}(\nu) \cap \pi'(E) \cap \mathrm{reg}(\mathrm{Sec}(Y)) \subseteq \mathrm{Dom}(\nu) \cap \pi'(E). \tag{5.20}$$

Now, consider the natural map $\rho : \mathrm{Hilb}(Y) \longrightarrow \mathbb{G}(9,26)$, defined by $\rho(Q) = \langle Q \rangle$ and the closed subset $C := \{(q,L) \in \mathbb{P}^{26} \times \mathbb{G}(9,26) : q \in L\}$ of $\mathbb{P}^{26} \times \mathbb{G}(9,26)$. We have

$$\begin{aligned} & \left((\mathrm{Id}_{\mathbb{P}^{26}} \times \rho)^{-1}(C) \cap \mathrm{Dom}(\nu) \times \mathrm{Hilb}(Y)\right) \bigcap \mathrm{Graph}(\nu : \mathrm{Dom}(\nu) \to \mathrm{Hilb}(Y)) \\ = \ & \{(q,Q) \in \mathrm{Dom}(\nu) \times \mathrm{Hilb}(Y) : Q = \nu(q) \text{ and } q \in \langle Q \rangle\} \\ \simeq \ & \{q \in \mathrm{Dom}(\nu) : q \in \langle \nu(q) \rangle\} =: T, \end{aligned}$$

from which it follows that the set $T$ is closed in $\mathrm{Dom}(\nu)$ and, since $\mathrm{Dom}(\nu) \supseteq T \supseteq V$, we have

$$\mathrm{Dom}(\nu) = \overline{T} = T = \{q \in \mathrm{Dom}(\nu) : q \in \langle \nu(q) \rangle\}. \tag{5.21}$$

By (5.20) and (5.21) it follows that, for the general point $p \in \pi'(E) \setminus Y$, we have $p \in \langle \nu(p) \rangle$, hence $\langle \nu(p) \rangle \subseteq L_p(Y)$ and then $\nu(p) = \Sigma_p(Y)$. Thus $\nu(\mathrm{Dom}(\nu))$ intersects the open $U(Y)$ of $\mathrm{Hilb}(Y)$ consisting of the smooth 8-dimensional quadrics contained in $Y$ and then, for a general point $q$ in the nonempty open set $\nu|_{\mathrm{Dom}(\nu)}^{-1}(U(Y))$, the entry locus $\Sigma_q(Y)$ is a smooth 8-dimensional quadric.

*Case* 5.19.2 (Suppose $\pi'(E) \subseteq \mathrm{sing}(\mathrm{Sec}(Y))$). [4] Let $z \in Y$ be a general point, $\tau_{z,Y} : Y \dashrightarrow W_{z,Y} \subset \mathbb{P}^9$ the tangential projection (note that $W_{z,Y}$ is a nonlinear hypersurface, by Proposition 1.14) and let $q \in \mathrm{Sec}(Y) \setminus Y$ be a general point. Arguing as in [CC10, Claim 8.8] we obtain that $T_z(Y) \cap L_q(Y) = T_z(Y) \cap \langle \Sigma_q(Y) \rangle = \emptyset$, so we deduce that $\tau_{z,Y}$ isomorphically maps $\Sigma_q(Y)$ to $W_{z,Y}$. From this and Terracini Lemma it follows that

$$\mathrm{Sec}(Y) = S(\Sigma_q(Y), Y) := \overline{\bigcup_{\substack{(z_1,z_2) \in \Sigma_q(Y) \times Y \\ z_1 \neq z_2}} \langle z_1, z_2 \rangle}. \tag{5.22}$$

Now suppose by contradiction that there exists $w \in \mathrm{Vert}(\Sigma_q(Y))$. By (5.22) it follows that $w \in \mathrm{Vert}(\mathrm{Sec}(Y))$ and by assumption we obtain $\pi'(E) \subseteq S(w, \pi'(E)) \subseteq \mathrm{sing}(\mathrm{Sec}(Y))$. Thus $w \in \mathrm{Vert}(\pi'(E))$ and hence $T_w(\pi'(E)) = \mathbb{P}^{26}$. This yields that $\mathbf{Q}$ is singular in $w$, a contradiction.

[4] Note that this is the only place in the proof where we use the smoothness of $\mathbf{Q}$. Note also that this case does not arise if one first shows that $\deg(\mathrm{Sec}(Y)) = 3$. Indeed, let $\mathscr{C} \subset \mathbb{P}^N$ be a cubic hypersurface and $D \subset \mathscr{C}$ an irreducible divisor contained in $\mathrm{sing}(\mathscr{C})$. Then, for a general plane $\mathbb{P}^2 \subset \mathbb{P}^N$, putting $C = \mathscr{C} \cap \mathbb{P}^2$ and $\Lambda = D \cap \mathbb{P}^2$, we have $\Lambda \subseteq \mathrm{sing}(C)$. Hence, by Bézout's Theorem, being $C$ a plane cubic curve, it follows $\deg(D) = \#(\Lambda) = 1$.

Finally, since for general points $z \in Y$ and $q \in \mathrm{Sec}(Y) \setminus Y$, we have $W_{z,Y} = \overline{\tau_z(Y)} \simeq \Sigma_q(Y)$ and $\Sigma_q(Y)$ is a smooth quadric, we deduce that $W_{z,Y}$ is smooth. It follows that the Gauss map $G_{W_{z,Y}} : W_{z,Y} \dashrightarrow \mathbb{G}(8,9) = \left(\mathbb{P}^9\right)^*$ is birational onto its image (see Corollary 1.13) and hence the dimension of its general fiber is $\widetilde{\gamma}(Y) = \gamma(Y) - \delta(Y) = 0$. □

Now, consider two general points $z_1, z_2 \in Y$ ($z_1 \neq z_2$), a general point $q \in \langle z_1, z_2 \rangle$ and put $\Sigma_q = \Sigma_q(Y)$, $L_q = L_q(Y)$, $H_q = T_q(\mathrm{Sec}(Y))$. Let $\pi_{L_q} : Y \dashrightarrow \mathbb{P}^{16}$ be the linear projection from $L_q$ and $H'_q \simeq \mathbb{P}^{25} \subset \mathbb{P}^{16}$ the projection of $H_q$ from $L_q$. We note that, by the proof of the Scorza Lemma in [Rus10], it follows that $\Sigma_q = \overline{\tau^{-1}_{z_1,Y}(\tau_{z_1,Y}(z_2))}$ and, since the image of the general tangential projection is smooth of dimension 8 and the general entry locus is smooth of dimension $8 = \dim(Y) - \dim(W_{z_1,Y})$, it follows that $Y$ is smooth along $\Sigma_q(Y)$. In particular, it follows that $H_q$ is tangent to $Y$ along $\Sigma_q$.

*Claim* 5.19.6. $\pi_{L_q}$ is birational.

*Proof of the Claim.* See also [Rus10, Proposition 3.3.14]. By the generality of the points $z_1, z_2, q$ it follows

$$L_q = \langle T_{z_1}(Y), z_2 \rangle \cap \langle T_{z_2}(Y), z_1 \rangle. \tag{5.23}$$

The projection from the linear space $\langle T_{z_1}(Y), z_2 \rangle$ can be obtained as the composition of the tangential projection $\tau_{z_1,Y} : Y \dashrightarrow W_{z_1,Y} \subset \mathbb{P}^9$ and of the projection of $W_{z_1,Y}$ from the point $\tau_{z_1,Y}(z_2)$. Thus the projection from $\langle T_{z_1}(Y), z_2 \rangle$, $\pi_{z_1,z_2} : Y \dashrightarrow \mathbb{P}^8$ is dominant and for the general point $z \in Y$ we get

$$\langle T_{z_1}(Y), z_2, z \rangle \cap Y \setminus \langle T_{z_1}(Y), z_2 \rangle = \pi^{-1}_{z_1,z_2}(\pi_{z_1,z_2}(z)) = Q_{z_1,z} \setminus \langle T_{z_1}(Y), z_2 \rangle, \tag{5.24}$$

where $Q_{z_1,z}$ denotes the entry locus of $Y$ with respect to a general point on $\langle z_1, z \rangle$. Similarly

$$\langle T_{z_2}(Y), z_1, z \rangle \cap Y \setminus \langle T_{z_2}(Y), z_1 \rangle = \pi^{-1}_{z_2,z_1}(\pi_{z_2,z_1}(z)) = Q_{z_2,z} \setminus \langle T_{z_2}(Y), z_1 \rangle. \tag{5.25}$$

Now $\pi^{-1}_{L_q}(\pi_{L_q}(z)) = \langle L_q, z \rangle \cap Y \setminus \Sigma_q$ and, by the generality of $z$,

$$\pi^{-1}_{L_q}(\pi_{L_q}(z)) = \langle L_q, z \rangle \cap Y \setminus H_q. \tag{5.26}$$

By (5.23), (5.24), (5.25) and (5.26) and observing that the spaces $\langle T_{z_1}(Y), z_2 \rangle$, $\langle T_{z_2}(Y), z_1 \rangle$ are contained in $H_q$, it follows

$$\{z\} \subseteq \pi^{-1}_{L_q}(\pi_{L_q}(z)) \subseteq Q_{z_1,z} \cap Q_{z_2,z}. \tag{5.27}$$

Finally, as we have already observed in Claim 5.19.5, the restriction of the tangential projection $\tau_{z_1,Y}$ to $Q_{z_2,z}$ is an isomorphism $\bar{\tau} := (\tau_{z_1,Y})|_{Q_{z_2,z}} : Q_{z_2,z} \to W_{z_1,Y}$; hence

$$\{z\} = \bar{\tau}^{-1}(\bar{\tau}(z)) = \tau^{-1}_{z_1,Y}(\tau_{z_1,Y}(z)) \cap Q_{z_2,z} = Q_{z_1,z} \cap Q_{z_2,z}. \tag{5.28}$$

By (5.27) and (5.28), it follows $\pi^{-1}_{L_q}(\pi_{L_q}(z)) = \{z\}$ and hence the birationality of $\pi_{L_q}$. □

*Claim* 5.19.7. $\pi_{L_q}$ induces an isomorphism $Y \setminus H_q \xrightarrow{\simeq} \mathbb{P}^{16} \setminus H'_q$.

*Proof of the Claim.* We resolve the indeterminacies of $\pi_{L_q}$ with the diagram

$$\begin{tikzcd} & \mathrm{Bl}_{\Sigma_q}(Y) \arrow[dl, "\alpha"'] \arrow[dr, "\widetilde{\pi_{L_q}}"] & \\ Y \arrow[rr, dashed, "\pi_{L_q}"] & & \mathbb{P}^{16} \end{tikzcd}$$

The morphism $\widetilde{\pi_{L_q}}$ is projective and birational and hence surjective. Moreover, the points of the base locus of $\pi_{L_q}^{-1}$ are the points for which the fiber of $\widetilde{\pi_{L_q}}$ has positive dimension. Since $H_q \supseteq L_q$ and $H_q \cap Y = \overline{\pi_{L_q}^{-1}(H_q')}$, in order to prove the assertion, it suffices to show that, for every $w \in \mathbb{P}^{16} \setminus H_q'$, $\dim\left(\widetilde{\pi_{L_q}}^{-1}(w)\right) = 0$. Suppose by contradiction that there exists $w \in \mathbb{P}^{16} \setminus H_q'$ such that $Z := \widetilde{\pi_{L_q}}^{-1}(w)$ has positive dimension. Then, for the choice of $H_q$, we have $\emptyset = Z \cap \alpha^{-1}(H_q \cap Y) \supseteq Z \cap \alpha^{-1}(\Sigma_q)$ and therefore $\alpha(Z)$ contains an irreducible curve $C$ with $\pi_{L_q}(C) = w$ and $C \cap L_q = \emptyset$, against the fact that a linear projection, when is defined everywhere, is a finite morphism. $\square$

*Claim* 5.19.8. $Y$ is smooth.

*Proof of the Claim.* Suppose that there exists a point $z_0$ with

$$z_0 \in \bigcap_{\substack{q \in \mathrm{Sec}(Y) \\ \text{generale}}} T_q(\mathrm{Sec}(Y)) \cap Y = \bigcap_{q \in \mathrm{Sec}(Y)} T_q(\mathrm{Sec}(Y)) \cap Y = \mathrm{Vert}(\mathrm{Sec}(Y)) \cap Y.$$

If $z \in Y$ is a general point, since $Y$ is not a cone, the tangential projection $\tau_{z,Y}$ is defined in $z_0$ and it follows that $\tau_{z,Y}(z_0)$ is a vertex of $W_{z,Y}$. This contradicts Claim 5.19.5 and hence we have $\bigcap_{\substack{q \in \mathrm{Sec}(Y) \\ \text{generale}}} T_q(\mathrm{Sec}(Y)) \cap Y = \emptyset$, from which we conclude by Claim 5.19.7. $\square$

Now we can conclude the proof of Theorem 5.19. By Claim 5.19.8 it follows that $Y \subset \mathbb{P}^{26}$ is a Severi variety, so by their classification (see Theorem 2.22) it follows that $Y = E_6$; moreover, since $27 = h^0(\mathbb{P}^{26}, \mathscr{I}_{\mathfrak{B}',\mathbb{P}^{26}}(2)) \leq h^0(\mathbb{P}^{26}, \mathscr{I}_{Y,\mathbb{P}^{26}}(2)) = 27$, we have $Y = \mathfrak{B}'$. Now, by the classification of the special Cremona transformations of type $(2,2)$ in Theorem 2.23 (or also by a direct calculation), it follows that the lifting $\psi : \mathbb{P}^{26} \dashrightarrow \mathbb{P}^{26}$ of $\varphi^{-1} : \mathbf{Q} \dashrightarrow \mathbb{P}^{25} =: H \subset \mathbb{P}^{26}$ is a birational transformation of type $(2,2)$ and therefore the base locus $\widehat{\mathfrak{B}} \subset \mathbb{P}^{26}$ of the inverse of $\psi$ is again the variety $E_6$. Of course $\widehat{\mathfrak{B}} \cap H = \mathfrak{B}$ and hence the thesis. $\square$

*Remark* 5.20. We observe that from Claim 5.19.5 and Proposition 2.25 it follows that $Y$ is a $R_1$-variety and hence by Theorem 2.26 it follows Claim 5.19.8.

## 5.5 Transformations whose base locus has dimension $\leq 3$

Let $\varphi$ be a special transformation as in Assumption 5.1 and let $r \leq 3$. From Proposition 5.2 we get the following possibilities for $(r,n)$: $(1,4)$; $(2,6)$; $(3,7)$; $(3,8)$. If $(r,n) \in \{(1,4),(3,7)\}$ then $(d,\Delta) = (2,2)$ and these cases have already been classified in Theorem 5.19.

### 5.5.1 Case $(r,n) = (2,6)$

**Proposition 5.21.** *Let $\varphi : \mathbb{P}^6 \dashrightarrow \overline{\varphi(\mathbb{P}^6)} = \mathbf{S} \subset \mathbb{P}^7$ be birational and special of type $(2,d)$, with $\mathbf{S}$ a factorial hypersurface of degree $\Delta \geq 2$. If $r = \dim(\mathfrak{B}) = 2$, then $\mathfrak{B}$ is the blow-up $\sigma : \mathrm{Bl}_{\{p_0,\ldots,p_5\}}(\mathbb{P}^2) \to \mathbb{P}^2$ of 6 points in the plane with $H_{\mathfrak{B}} \sim \sigma^*(4H_{\mathbb{P}^2}) - 2E_0 - E_1 - \cdots - E_5$ ($E_0,\ldots,E_5$ are the exceptional divisors). Moreover, we have $d = 3$ and $\Delta = 2$.*

*Proof.* By Lemma 5.4, it follows $\chi(\mathfrak{B},\mathscr{O}_{\mathfrak{B}}(1)) = 7$ and $\chi(\mathfrak{B},\mathscr{O}_{\mathfrak{B}}(2)) = 20$, from which we deduce

$$P_{\mathfrak{B}}(t) = \left(\lambda\, t^2 + (26-3\lambda)\, t + 2\lambda - 12\right)/2 = \left((g+12)\, t^2 + (16-3g)\, t + 2g\right)/4$$

and hence $g = 2(\lambda-6)$. In particular $\lambda \geq 6$, being $g \geq 0$. Now, if $\lambda \leq 2\,\mathrm{codim}_{\mathbb{P}^6}(\mathfrak{B}) + 1 = 9$, cutting $\mathfrak{B}$ with a general $\mathbb{P}^4 \subset \mathbb{P}^6$, we obtain a set $\Lambda \subset \mathbb{P}^4$ of $\lambda$ points that imposes independent conditions to the quadrics of $\mathbb{P}^4$ (Lemma 3.2); hence $h^0(\mathbb{P}^6,\mathscr{I}_{\mathfrak{B},\mathbb{P}^6}(2)) \leq h^0(\mathbb{P}^4,\mathscr{I}_{\Lambda,\mathbb{P}^4}(2)) = h^0(\mathbb{P}^4,\mathscr{O}_{\mathbb{P}^4}(2)) - \lambda$, i.e. $\lambda \leq 7$. Moreover, if $\lambda \geq 9$, $\Lambda$ would impose at least 9 conditions to the quadrics and hence we would get the contradiction $h^0(\mathbb{P}^6,\mathscr{I}_{\mathfrak{B}}(2)) \leq 6$. Hence $\lambda = 6$ or $\lambda = 7$, and in both cases, knowing the expression of the Hilbert polynomial, we conclude applying [Ion84]: if $\lambda = 6$, such a variety does not exist; if $\lambda = 7$, then $\mathfrak{B}$ is as asserted. Finally, by Remark 5.3, we get either $(d,\Delta) = (3,2)$ or $(d,\Delta) = (2,3)$, but the latter case is impossible by Example 5.14 (the pair $(d,\Delta)$ may also be determined by calculating the Chern classes of $\mathfrak{B}$). $\square$

### 5.5.2 Case $(r,n) = (3,8)$

Firstly we observe that if $(r,n) = (3,8)$ by Remark 5.3 it follows $d + \Delta = 6$ and hence we have $(d,\Delta) \in \{(2,4),(3,3),(4,2)\}$.

*Remark* 5.22. See also Chap. 6, p. 62. Let notation be as in the proof of Proposition 5.2 and let $(r,n) = (3,8)$. Denote by $c_j := c_j(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}}^{3-j}$ (resp. $s_j := s_j(\mathscr{N}_{\mathfrak{B},\mathbb{P}^8}) \cdot H_{\mathfrak{B}}^{3-j}$), for $1 \leq j \leq 3$, the degree of the $j$-th Chern class (resp. Segre class) of $\mathfrak{B}$. From the exact sequence $0 \to \mathscr{T}_{\mathfrak{B}} \to \mathscr{T}_{\mathbb{P}^8}|_{\mathfrak{B}} \to \mathscr{N}_{\mathfrak{B},\mathbb{P}^8} \to 0$ we get: $s_1 = c_1 - 9\lambda$, $s_2 = c_2 - 9c_1 + 45\lambda$, $s_3 = c_3 - 9c_2 + 45c_1 - 165\lambda$. Moreover

$$\begin{aligned}
\lambda &= H_{\mathfrak{B}}^3 = -K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 + 2g - 2 - \lambda = s_1 + 8\lambda + 2g - 2,\\
d\Delta &= dH'^8 = H'^7 \cdot (dH' - E') = (2H - E)^7 \cdot H\\
&= -H \cdot E^7 + 14H^2 \cdot E^6 - 84H^3 \cdot E^5 + 128H^8 = -s_2 - 14s_1 - 84\lambda + 128,\\
\Delta &= H'^8 = (2H-E)^8 = E^8 - 16H \cdot E^7 + 112H^2 \cdot E^6 - 448H^3 \cdot E^5 + 256H^8\\
&= -s_3 - 16s_2 - 112s_1 - 448\lambda + 256,
\end{aligned}$$

and hence

$$\begin{cases} s_1 = -7\lambda - 2g + 2, \\ s_2 = 14\lambda + 28g - d\Delta + 100, \\ s_3 = 112\lambda - 224g + (16d-1)\Delta - 1568, \end{cases} \qquad \begin{cases} c_1 = 2\lambda - 2g + 2, \\ c_2 = -13\lambda + 10g - d\Delta + 118, \\ c_3 = 70\lambda - 44g + (7d-1)\Delta - 596. \end{cases}$$

**Proposition 5.23.** *Let $\varphi : \mathbb{P}^8 \dashrightarrow \overline{\varphi(\mathbb{P}^8)} = \mathbf{S} \subset \mathbb{P}^9$ be birational and special of type $(2,d)$, with $\mathbf{S}$ a factorial hypersurface of degree $\Delta \geq 2$. If $r = \dim(\mathfrak{B}) = 3$, then one of the following cases holds:*

*(i)* $\lambda = 12$, $g = 7$, $d = 4$, $\Delta = 2$, *$\mathfrak{B}$ is a linear section of the spinorial variety $S^{10} \subset \mathbb{P}^{15}$;*

*(ii)* $\lambda = 12$, $g = 7$, $d = 2$, $\Delta = 4$, *$\mathfrak{B}$ is a Mukai variety with Betti numbers $b_2 = 2$, $b_3 = 18$;*

*(iii)* $\lambda = 11$, $g = 5$, $d = 3$, $\Delta = 3$, *$\mathfrak{B}$ is the variety $\mathfrak{Q}_{p_1 \ldots, p_5}$ defined as the blow-up of 5 points $p_1, \ldots, p_5$ (possibly infinitely near) in a smooth quadric $Q \subset \mathbb{P}^4$, with $H_{\mathfrak{Q}_{p_1,\ldots,p_5}} \sim \sigma^*(2H_{\mathbb{P}^4}|_Q) - E_1 - \cdots - E_5$, where $\sigma$ is the blow-up map and $E_1, \ldots, E_5$ are the exceptional divisors;*

*(iv)* $\lambda = 11$, $g = 5$, $d = 4$, $\Delta = 2$, *$\mathfrak{B}$ is a scroll over $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O} \oplus \mathscr{O}(-1))$.*

*Proof.* By Proposition 5.5, $\mathfrak{B} \subset \mathbb{P}^8$ is nondegenerate and linearly normal. Let $\Lambda \subset C \subset S \subset \mathfrak{B} \subset \mathbb{P}^8$ be a sequence of general linear sections of $\mathfrak{B}$. By the exact sequence $0 \to \mathscr{I}_{\mathfrak{B},\mathbb{P}^8}(-1) \to \mathscr{I}_{\mathfrak{B},\mathbb{P}^8} \to \mathscr{I}_{S,\mathbb{P}^7} \to 0$ and those similar for $S$ and $C$, using that $\mathfrak{B}$, $S$, $C$ are nondegenerate, we get the inequality $h^0(\mathbb{P}^8, \mathscr{I}_{\mathfrak{B},\mathbb{P}^8}(2)) \leq h^0(\mathbb{P}^5, \mathscr{I}_{\Lambda,\mathbb{P}^5}(2))$. In particular, putting $h_\Lambda(2) := \dim(\mathrm{Im}(H^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2)) \to H^0(\Lambda, \mathscr{O}_\Lambda(2))))$, we have

$$h_\Lambda(2) \leq h^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2)) - h^0(\mathbb{P}^8, \mathscr{I}_{\mathfrak{B},\mathbb{P}^8}(2)) = 11. \tag{5.29}$$

If now $\#(\Lambda) = \lambda \geq 11$, taking $\Lambda' \subseteq \Lambda$ with $\#(\Lambda') = 11$, by Lemma 3.2 we obtain

$$h_\Lambda(2) \geq h^0(\mathbb{P}^5, \mathscr{O}_{\mathbb{P}^5}(2)) - h^0(\mathbb{P}^5, \mathscr{I}_{\Lambda',\mathbb{P}^5}(2)) = \#(\Lambda') = 11. \tag{5.30}$$

The inequalities (5.29) and (5.30) yield $h_\Lambda(2) = 2 \cdot 6 - 1$ and this, by Proposition 3.3, yields a contradiction if $\lambda \geq 13$. Thus we have $\lambda \leq 12$ and, by Castelnuovo's bound (Proposition 3.1), we also have

$$K_S \cdot H_S = (K_{\mathfrak{B}} + H_{\mathfrak{B}}) \cdot H_{\mathfrak{B}}^2 = 2g - 2 - \lambda \leq 0. \tag{5.31}$$

We discuss two cases.

*Case* 5.23.1 (Suppose $K_S \nsim 0$). By (5.31) and by the proof of [Har77, V Lemma 1.7], it follows that $h^2(S, \mathscr{O}_S) = h^2(S, \mathscr{O}_S(1)) = 0$. Consequently, by Lemma 5.4 and by the exact sequence $0 \to \mathscr{O}_{\mathfrak{B}}(-1) \to \mathscr{O}_{\mathfrak{B}} \to \mathscr{O}_S \to 0$, we obtain

$$h^2(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) = h^3(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) = h^3(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(-1)) = 0.$$

Moreover $h^1(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) = h^1(S, \mathscr{O}_S) =: q$ and using again Lemma 5.4 we obtain

$$\begin{array}{ll} \chi(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(-1)) = 0, & \chi(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) = 1 - q, \\ \chi(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(1)) = 9, & \chi(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(2)) = 35. \end{array} \tag{5.32}$$

Now, the conditions (5.32) determine $P_{\mathfrak{B}}(t)$ in function of $q$, from which in particular we obtain $\lambda = 11 - 3q$, $g = 5 - 5q$. Being $g \geq 0$, we have $(q, \lambda, g) = (0, 11, 5)$ or $(q, \lambda, g) = (1, 8, 0)$, but the latter case is impossible by [Fuj90, Theorems 10.2 and 12.1, Remark 12.2]. Thus we have

$$q = 0,\ P_{\mathfrak{B}}(t) = \left(11t^3 + 21t^2 + 16t + 6\right)/6,\ K_S \cdot H_S = -3,\ g = 5. \tag{5.33}$$

Applying the main result in [Ion85] and the numerical constraints in [BB05a] and [BB05b], it follows immediately that $\mathfrak{B}$ is one of the following:

(a) the variety $\mathfrak{Q}_{p_1,\ldots,p_5}$;

(b) a scroll over a surface $Y$, where $Y$ is either the blow-up of 5 points in $\mathbb{P}^2$, or the rational ruled surface $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}\oplus\mathscr{O}(-1))$;

(c) a quadric fibration over $\mathbb{P}^1$.

Now, if $\mathfrak{B}$ is as in case (b), we use the well-known relation (multiplicativity of the topological Euler characteristic) $c_3(\mathfrak{B}) = c_1(\mathbb{P}^1)c_2(Y)$, and by Remark 5.22 we deduce $\Delta = (2c_2(Y)+46)/(7d-1)$. Moreover, if $Y$ is the blow-up of 5 points in $\mathbb{P}^2$, we have $c_2(Y) = 12\chi(\mathscr{O}_Y) - K_Y^2 = 12\chi(\mathscr{O}_{\mathbb{P}^2}) - (K_{\mathbb{P}^2}^2 - 5) = 8$, while if $Y$ is $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}\oplus\mathscr{O}(-1))$, we have $c_2(Y) = 4$. Thus, if $\mathfrak{B}$ is as in case (b), we have $Y = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}\oplus\mathscr{O}(-1))$, $d = 4$ and $\Delta = 2$. If $\mathfrak{B}$ is as in case (c), we easily deduce that $c_2(\mathfrak{B}) = 20$ and hence, again by Remark 5.22, we obtain the contradiction $d\Delta = 5$. Now suppose $\mathfrak{B}$ as in case (a), namely $\mathfrak{B}$ is obtained as a sequence

$$\mathfrak{B} = Z_5 \xrightarrow{\sigma_5} Z_4 \xrightarrow{\sigma_4} \cdots \xrightarrow{\sigma_1} Z_0 = Q,$$

where $\sigma_j$ is the blow-up at a point $p_j \in Z_{j-1}$, $H_{Z_j} = \sigma_j^*(H_{Z_{j-1}}) - E_j$, $E_j$ is the exceptional divisor and $H_{Z_0} = H_Q = 2H_{\mathbb{P}^4}|_Q$. By [GH78, page 609] it follows that $c_2(Z_j) = \sigma_j^*(c_2(Z_{j-1}))$ and hence

$$c_2(Z_j)\cdot H_{Z_j} = \sigma_j^*(c_2(Z_{j-1}))\cdot\sigma_j^*(H_{Z_{j-1}}) - \sigma_j^*(c_2(Z_{j-1}))\cdot E_j = c_2(Z_{j-1})\cdot H_{Z_{j-1}}.$$

In particular, we obtain $c_2(\mathfrak{B})\cdot H_{\mathfrak{B}} = 2c_2(Q)\cdot H_{\mathbb{P}^4}|_Q = 16$. On the other hand, by Remark 5.22, we obtain that $c_2(\mathfrak{B})\cdot H_{\mathfrak{B}} = 25 - d\Delta$, hence $d\Delta = 9$.

*Case* 5.23.2 (Suppose $K_S \sim 0$). By Castelnuovo's bound, since $K_S\cdot H_S = 0$, it follows that $(\lambda, g) = (12, 7)$ and hence also that $\chi(S,\mathscr{O}_S) = -3\lambda + 2g + 24 = 2$ (note that, as in the previous case, we know the values of $P_{\mathfrak{B}}(1)$ and $P_{\mathfrak{B}}(2)$). We have $q = h^1(S,\mathscr{O}_S) = 1 - \chi(S,\mathscr{O}_S) + h^2(S,\mathscr{O}_S) = -1 + h^2(S,K_S) = 0$ and hence $S$ is a $K3$-surface, $C$ is a canonical curve and $\mathfrak{B}$ is a Mukai variety. Now we denote by $b_j = b_j(\mathfrak{B})$ the $j$-th Betti number of $\mathfrak{B}$. By Poincaré-Hopf index formula and Poincaré duality (see for example [GH78]) we have $c_3(\mathfrak{B}) = \sum_j(-1)^j b_j = 2 + 2b_2 - b_3$ and, by Remark 5.22, we also have $c_3(\mathfrak{B}) = -7d^2 + 43d - 70$. Moreover, by [MM81], if $b_2 \geq 2$ then $(b_2, b_3) \in \{(2,12),(2,18),(3,16),(9,0)\}$. Thus, if $b_2 \geq 2$ we have $b_2 = 2$, $b_3 = 18$, $d = 2$, $\Delta = 4$. Finally, by Theorem 2.8, if $b_2 = 1$ then $\mathfrak{B}$ is a linear section of the spinorial variety $S^{10} \subset \mathbb{P}^{15}$. Thus, we have a natural inclusion $\iota : H^0(\mathbb{P}^{15},\mathscr{I}_{S^{10}}(2)) \hookrightarrow H^0(\mathbb{P}^8,\mathscr{I}_{\mathfrak{B}}(2))$ and, since $h^0(\mathbb{P}^{15},\mathscr{I}_{S^{10}}(2)) = h^0(\mathbb{P}^8,\mathscr{I}_{\mathfrak{B}}(2))$, we see that $\iota$ is an isomorphism. This says that $\varphi$ is the restriction of the map $\psi : \mathbb{P}^{15} \dashrightarrow \mathbf{Q} \subset \mathbb{P}^9$ given in Example 5.16.

□

*Remark* 5.24 (on case (iv) of Proposition 5.23). More precisely, from [BB05a, Proposition 4.2.3] it follows that $\mathfrak{B} = \mathbb{P}_{\mathbb{F}_1}(\mathscr{E})$, where $(\mathbb{F}_1, H_{\mathbb{F}_1}) := (\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}\oplus\mathscr{O}(-1)), C_0 + 2f)$ (notation as in [Har77, page 373]) and $\mathscr{E}$ is a locally free sheaf of rank 2 on $\mathbb{F}_1$, with $c_2(\mathscr{E}) = 10$. $\mathbb{F}_1$ is thus the cubic surface of $\mathbb{P}^4$ with ideal generated by: $x_0x_3 - x_2x_1, x_0x_4 - x_3x_1, x_2x_4 - x_3^2$ and it is isomorphic to $\mathbb{P}^2$ with one point blown up. We point out that the problem of the existence of an example for case (iv) of Proposition 5.23 is essentially reduced to showing that such a scroll over $\mathbb{F}_1$ must be cut out by quadrics. For further details we refer to Example 6.39 below.

**Proposition 5.25.** *Let $\varphi : \mathbb{P}^8 \dashrightarrow \overline{\varphi(\mathbb{P}^8)} = \mathbf{S} \subset \mathbb{P}^9$ be birational and special of type $(2,d)$, with $\mathbf{S}$ a hypersurface as in Assumption 5.1. If $r = \dim(\mathfrak{B}) = 3$, then either case (i), case (iii), or case (iv) of Proposition 5.23 holds.*

*First proof of Proposition 5.25.* We have to exclude case (ii) of proposition 5.23, so we just assume that $\mathfrak{B}$ is as in this case. Using the fact that $K_{\mathfrak{B}} \sim -H_{\mathfrak{B}}$ and $(\lambda, g) = (12, 7)$ we can compute the Segre classes of the tangent bundle of $\mathfrak{B}$:

$$\begin{aligned} s_1(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}}^2 &= -\lambda = -12, \\ s_2(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}} &= -24 + \lambda = -12, \\ s_3(\mathscr{T}_{\mathfrak{B}}) &= -c_3(\mathfrak{B}) + 48 - \lambda = 100 - (7d-1)\Delta. \end{aligned}$$

Since $\mathfrak{B}$ is a *QEL*-variety of type $\delta = 0$ we apply the *double point formula* (see for example [PS76] and [Lak76] and also Proposition 6.11 below)

$$2(2d-1) = 2\deg(\mathrm{Sec}(\mathfrak{B})) = \lambda^2 - \sum_{j=0}^{3} \binom{7}{j} s_{3-j}(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}}^j = (7d-1)\Delta - 40, \tag{5.34}$$

from which we deduce $\Delta = (4d+38)/(7d-1)$ i.e. $d = 4$ and $\Delta = 2$. □

*Second proof of Proposition 5.25.* Let $x \in \mathfrak{B}$ be a general point and put $k = \#(\mathscr{L}_{x,\mathfrak{B}})$. Since the variety $\mathfrak{B}$ is not a scroll over a curve (otherwise it would happen $\lambda^2 \geq (2r+1)\lambda + r(r+1)(g-1)$, by [BB05b]) and it is defined by quadrics, by [CMR04, Proposition 5.2], it follows that the support of the base locus of the tangential projection $\tau_{x,\mathfrak{B}} : \mathfrak{B} \dashrightarrow W_{x,\mathfrak{B}} \subset \mathbb{P}^4$, i.e. $(T_x(\mathfrak{B}) \cap \mathfrak{B})_{\mathrm{red}}$, consists of $0 \leq k < \infty$ lines through $x$. Now, by Proposition 5.2, $\mathfrak{B}$ is a *QEL*-variety of type $\delta = 0$ and repeating the argument in [CMR04, §5] (keeping also in mind Theorem 2.15) we get the relation $\lambda - 8 + k = \deg(W_{x,\mathfrak{B}})$. On the other hand, by proceeding as in Claim 5.19.2 or in [MR05, Theorem 1.4], we also obtain $\deg(W_{x,\mathfrak{B}}) \leq d$. Hence, we deduce

$$\lambda - 8 + k \leq d, \tag{5.35}$$

from which the conclusion follows. □

*Remark* 5.26. Note that in case (iii) of Proposition 5.23, by (5.35) it follows that $k = \#(\mathscr{L}_{x,\mathfrak{B}}) = 0$. We show directly that for a general point $x \in \mathfrak{Q} = \mathfrak{Q}_{p_1,\ldots,p_5}$, we have $\mathscr{L}_{x,\mathfrak{Q}} = \emptyset$. Suppose by contradiction that there exists $[l] \in \mathscr{L}_{x,\mathfrak{Q}}$. Then, by (1.5), $0 = \dim_{[l]}(\mathscr{L}_{x,\mathfrak{Q}}) = -K_{\mathfrak{Q}} \cdot l - 2$ and hence

$$(K_{\mathfrak{Q}} + 2H_{\mathfrak{Q}}) \cdot l = 0. \tag{5.36}$$

Moreover, by [Ion85, §0.3], the adjunction map $\psi_{\mathfrak{Q}}$, i.e. the map defined by the complete linear system $|K_{\mathfrak{Q}} + 2H_{\mathfrak{Q}}|$, is everywhere defined and we have a commutative diagram of adjunction maps

$$\begin{tikzcd} (\mathfrak{Q}, H_{\mathfrak{Q}}) \arrow[r, "\psi_{\mathfrak{Q}}"] \arrow[d, "\sigma"'] & \psi_{\mathfrak{Q}}(\mathfrak{Q}) \\ (Q, H_Q) \arrow[ur, "\psi_Q"'] & \end{tikzcd}$$

where $\sigma$ is the blow-up map and $(Q, H_Q) = (Q^3 \subset \mathbb{P}^4, 2H_{\mathbb{P}^4}|_Q)$. Now $K_Q + 2H_Q \sim (K_{\mathbb{P}^4} + Q)|_Q + 2(2H_{\mathbb{P}^4})|_Q \sim (-5H_{\mathbb{P}^4} + 2H_{\mathbb{P}^4} + 4H_{\mathbb{P}^4})|_Q \sim H_{\mathbb{P}^4}|_Q$, but this is in contradiction with (5.36).

### 5.5.3 Summary results

**Theorem 5.27.** *Table 5.5 classifies all special quadratic birational transformations as in Assumption 5.1 and with $r \leq 3$.*

| $r$ | $n$ | $\Delta$ | $d$ | $\delta$ | $\lambda$ | Abstract structure of $\mathfrak{B}$ | Examples |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 2 | 2 | 0 | 4 | $(\mathbb{P}^1, \mathscr{O}(4))$ | exist |
| 2 | 6 | 2 | 3 | 0 | 7 | Hyperplane section of an Edge variety | exist |
| 3 | 7 | 2 | 2 | 1 | 6 | Hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ | exist |
| 3 | 8 | 2 | 4 | 0 | 12 | Linear section of $S^{10} \subset \mathbb{P}^{15}$ | exist |
| 3 | 8 | 3 | 3 | 0 | 11 | $\mathfrak{Q}_{p_1,\ldots,p_5}$ | exist |
| 3 | 8 | 2 | 4 | 0 | 11 | Scroll over $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O} \oplus \mathscr{O}(-1))$ | not know |

Table 5.5: All transformations $\varphi$ as in Assumption 5.1 and with $r \leq 3$.

**Corollary 5.28.** *Let $\varphi$ be of type $(2,3)$ and let $\Delta = 3$. Then $\mathfrak{B}$ is the variety $\mathfrak{Q}_{p_1,\ldots,p_5} \subset \mathbb{P}^8$.*

*Proof.* By Propositions 5.2 and 5.8, $\mathfrak{B}$ is a *QEL*-variety of type $\delta = 0$ and dimension 3 in $\mathbb{P}^8$. Hence we apply Proposition 5.25. $\square$

In Chap. 6, we will extend these results in the case where **S** is not necessarily a hypersurface.

## 5.6 Invariants of transformations of type $(2,2)$ into a cubic and a quartic

*Remark* 5.29. Let $\delta \geq 3$ and consider $\mathscr{L}_{x,\mathfrak{B}} \subset \mathbb{P}^{r-1}$, where $x \in \mathfrak{B}$ is a general point. By Theorem 2.13 and Proposition 1.19 it follows that $\mathscr{L}_{x,\mathfrak{B}}$ is a smooth irreducible nondegenerate variety of codimension $(r - \delta + 2)/2$ and it is scheme-theoretic intersection of quadrics. Then, applying [BEL91, Corollary 2], we get that $\mathscr{L}_{x,\mathfrak{B}}$ is linearly normal.

In Propositions 5.30 and 5.31 we write $P = a_0, a_1, \ldots, a_r$ to indicate that $P_{\mathfrak{B}}(t) = a_0\binom{t}{r} + a_1\binom{t}{r-1} + \cdots + a_r$.

**Proposition 5.30.** *Let $\varphi$ be of type $(2,2)$ and let $\Delta = 3$. Then $\mathfrak{B}$ is a QEL-variety of type $\delta$ and a Fano variety of the first species of coindex c, as one of the following cases:*

*(i)* $n = 18$, $r = 10$, $\delta = 4$, $c = 4$, $P = 34, 272, 964, 1988, 2633, 2330, 1387, 544, 133, 18, 1$*; for the general point $x \in \mathfrak{B}$, $\mathscr{L}_{x,\mathfrak{B}} \subset \mathbb{P}^9$ is projectively equivalent to $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$.*

*(ii)* $n = 24$, $r = 14$, $\delta = 6$, $c = 5$, $P = 80, 920, 4866, 15673, 34302, 53884, 62541, 54366, 35472, 17228, 6104, 1521, 250, 24, 1$*; for the general point $x \in \mathfrak{B}$, $\mathscr{L}_{x,\mathfrak{B}} \subset \mathbb{P}^{13}$ is projectively equivalent to a smooth* 8*-dimensional linear section of $S^{10} \subset \mathbb{P}^{15}$.*

*Proof.* By Proposition 5.2 and Proposition 5.5 parts 1 and 2, and applying Theorems 2.19 and 2.14, it follows that $\mathfrak{B}$ is a *QEL*-variety of type $\delta$ and

$$(n,r,\delta) \in \{(6,2,0),(12,6,2),(18,10,4),(24,14,6)\}.$$

The tern $(6,2,0)$ is excluded by Proposition 5.21; the tern $(12,6,2)$ is excluded since otherwise by Proposition 5.5 part 3, we would get incompatible conditions for $P_{\mathfrak{B}}(t)$. The statement on $\mathscr{L}_{x,\mathfrak{B}}$, in the case (i) follows from Theorem 2.19, while in the case (ii) it follows from Theorem 2.8 or Theorem 2.18. $\square$

**Proposition 5.31.** *Let $\varphi$ be of type $(2,2)$ and let $\Delta = 4$. Then $\mathfrak{B}$ is a QEL-variety of type $\delta$ and a Fano variety of the first species of coindex c, as one of the following cases:*

*(i)* $n = 17$, $r = 9$, $\delta = 3$, $c = 4$, $P =$ 35, 245, 747, 1297, 1406, 980, 435, 117, 17, 1*; for the general point $x \in \mathfrak{B}$, $\mathscr{L}_{x,\mathfrak{B}} \subset \mathbb{P}^8$ is projectively equivalent to* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(1) \oplus \mathscr{O}(1) \oplus \mathscr{O}(2)) \subset \mathbb{P}^8$.

*(ii)* $n = 23$, $r = 13$, $\delta = 5$, $c = 5$, $P =$ 82, 861, 4126, 11932, 23195, 31943, 31984, 23504, 12628, 4875, 1306, 228, 23, 1*; for the general point $x \in \mathfrak{B}$, $\mathscr{L}_{x,\mathfrak{B}} \subset \mathbb{P}^{12}$ is projectively equivalent to a smooth* 7*-dimensional linear section of $S^{10} \subset \mathbb{P}^{15}$.*

*Proof.* As in the proof of Proposition 5.30, we get that $\mathfrak{B}$ is a *QEL*-variety of type $\delta$ and dimension $r$ with

$$(n,r,\delta) \in \{(8,3,0),(11,5,1),(17,9,3),(23,13,5)\}.$$

The case with $\delta = 0$ is excluded by Proposition 5.25; the case with $\delta = 1$ is excluded by Proposition 5.5 part 3; by the same Proposition, we get the expression of the Hilbert polynomials in the cases with $\delta \geq 3$. Finally, the statement on $\mathscr{L}_{x,\mathfrak{B}}$, in the case (i) follows from Theorem 2.19, while in the case (ii) it follows from Theorem 2.8 (for the latter case, by Kodaira Vanishing Theorem and Serre Duality, we get $g(\mathscr{L}_{x,\mathfrak{B}}) = 7$). $\square$

# Appendix A

# Further remarks on quadro-quadric birational transformations into a quadric

In this short appendix we provide a few more arguments to show Theorem 5.19, under an additional assumption.

Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{Q} \subset \mathbb{P}^{n+1}$ be a birational transformation of type $(2,2)$ into a smooth quadric $\mathbf{Q}$ and let $\mathfrak{B} \subset \mathbb{P}^n$ and $\mathfrak{B}' \subset \mathbf{Q} \subset \mathbb{P}^{n+1}$ be respectively the base locus of $\varphi$ and $\varphi^{-1}$.

**Definition A.1.** We shall say that $\varphi$ is *liftable* if there exists a Cremona transformation $\widehat{\varphi} : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n+1}$ of type $(2,2)$ such that $\varphi$ is the restriction of $\widehat{\varphi}$ to a hyperplane and therefore $\mathfrak{B}$ is a hyperplane section of the base locus $\widehat{\mathfrak{B}}$ of $\widehat{\varphi}$.

From Propositions 4.6 and 4.8 and by straightforward calculations follows that if $\mathfrak{B}$ is reduced and $n \leq 4$, then $\varphi$ is liftable; further, from Theorem 5.19 it follows that $\varphi$ is liftable whenever it is special. So we feel motivated to make the following:

**Conjecture A.2.** Each birational transformation $\mathbb{P}^n \dashrightarrow \mathbf{Q} \subset \mathbb{P}^{n+1}$ of type $(2,2)$ into a smooth quadric $\mathbf{Q}$ is liftable.

Recall that in [PR11] it was proved that *each Cremona transformation of type* $(2,2)$ *is, modulo changes of coordinates in the source and target space, an involution which is the adjoint of a rank* 3 *Jordan algebra.*

This has strong consequences in the event that Conjecture A.2 is true. For example, it implies that for every $\varphi$ as above, $\mathfrak{B}'$ can be considered projectively equivalent to $\widehat{\mathfrak{B}}$ and hence $\mathfrak{B}$ can be considered as a hyperplane section of $\mathfrak{B}'$. Further, if $\varphi$ is special, we have that $\widehat{\mathfrak{B}}$ is irreducible and $\mathrm{sing}(\widehat{\mathfrak{B}})$ is a finite set.[1] Then, restricting $\widehat{\varphi}$ to a general hyperplane, we get a special quadratic birational transformation into a general quadric containing $\mathfrak{B}'$. So we can treat $\mathfrak{B}$ like a general hyperplane section of $\widehat{\mathfrak{B}}$.

[1] Otherwise the very ample divisor $\mathfrak{B}$ of $\widehat{\mathfrak{B}}$ would intersect an irreducible curve contained in $\mathrm{sing}(\widehat{\mathfrak{B}})$ and then $\mathfrak{B}$ would be singular.

From these remarks it follows that the proof of Theorem 5.19 could be simplified if one had shown a priori that Conjecture A.2 holds in the special case. In fact, if it were the case, one could deduce that $\widetilde{\gamma}(\mathfrak{B}) \geq \widetilde{\gamma}(\widehat{\mathfrak{B}}) \geq 0$ and, as in Claim 5.19.3, one would deduce that $\widetilde{\gamma}(\mathfrak{B}) = 0$. Hence $\widehat{\mathfrak{B}}$ would be a Severi variety, by Theorem 2.26.

Assuming again that Conjecture A.2 holds in the special case, we can deduce Theorem 5.19 also by Proposition A.4, below. In fact, cases with $\delta \neq 7$ are clear and we can assume $\delta = 7$ and $\varphi : \mathbb{P}^{25} \dashrightarrow \mathbb{P}^{26}$; thus we obtain $\widehat{\mathfrak{B}}_{\mathrm{red}} = E_6 \subset \mathbb{P}^{26}$ and then $\widehat{\mathfrak{B}} = E_6$, since $h^0(\mathbb{P}^{26}, \mathscr{I}_{\widehat{\mathfrak{B}},\mathbb{P}^{26}}(2)) = h^0(\mathbb{P}^{26}, \mathscr{I}_{E_6,\mathbb{P}^{26}}(2))$.

*Remark* A.3. Let $Y \subset \mathbb{P}^n$ be a smooth *LQEL*-variety, with $\dim(Y) = 15$, $\dim(\mathrm{Sec}(Y)) = 24 < n$ and $\delta(Y) = 7$. Let $\widetilde{Y} \subset \mathbb{P}(H^0(Y, \mathscr{O}_Y(1))) = \mathbb{P}^{\widetilde{n}}$ be the linear normalization of $Y$. We have $\dim(\widetilde{Y}) = \dim(Y)$, $\dim(\mathrm{Sec}(\widetilde{Y})) = \dim(\mathrm{Sec}(Y))$, $\delta(\widetilde{Y}) = \delta(Y)$ and $\widetilde{Y}$ is an *LQEL*-variety. As in Claim 5.19.3 we obtain that, for the general point $x \in \widetilde{Y}$, $\mathscr{L}_{x,\widetilde{Y}} \subset \mathbb{P}^{14}$ is a hyperplane section of the spinorial variety $S^{10} \subset \mathbb{P}^{15}$; so, by Propositions 1.16 and 1.19, we deduce that $\widetilde{n} - 16 = \dim(|II_{x,\widetilde{Y}}|) \leq h^0(\mathbb{P}^{15}, \mathscr{I}_{S^{10},\mathbb{P}^{15}}(2)) - 1 = 9$, from which $\widetilde{n} \leq 25$. Thus $n = \widetilde{n} = 25$, $Y = \widetilde{Y}$ and $|II_{x,Y}|$ coincides with the second fundamental form of a hyperplane section of the Cartan variety $E_6 \subset \mathbb{P}^{26}$.

**Proposition A.4.** *Let $X \subset \mathbb{P}^{26}$ be an irreducible closed subscheme with* $\mathrm{Vert}(X) = \emptyset$. *Suppose that the intersection $Y = X \cap H \subset H$ of $X$ with a general hyperplane $H \subset \mathbb{P}^{26}$ is a smooth nondegenerate LQEL-variety of dimension* 15 *and of type $\delta = 7$. Then $X_{\mathrm{red}} = E_6 \subset \mathbb{P}^{26}$.*

*Proof.* Since $X \cap H = (X_{\mathrm{red}} \cap H)_{\mathrm{red}} = X_{\mathrm{red}} \cap H$ and $\mathrm{Vert}(X_{\mathrm{red}}) \subseteq \mathrm{Vert}(X)$, we can reduce to the case where $X$ is a reduced scheme. Hence $X \subset \mathbb{P}^{26}$ is an irreducible nondegenerate 16-dimensional variety which is singular at most in finitely many points, and $\mathrm{Sec}(X)$ is a nonlinear hypersurface with $\mathrm{Sec}(X) \cap H = \mathrm{Sec}(Y)$. For $x \in \mathrm{reg}(X)$, put $S_x = \{[L] \in \mathscr{L}_{x,X} : L \cap \mathrm{sing}(X) \neq \emptyset\}$ and consider the set $U = \{x \in \mathrm{reg}(X) : S_x = \emptyset\}$. We have

$$\begin{aligned} \mathrm{reg}(X) \setminus U &= \{x \in \mathrm{reg}(X) : \exists\, p \in \mathrm{sing}(X) \text{ with } \langle x, p \rangle \subseteq X\} \\ &= \mathrm{reg}(X) \cap \bigcup_{p \in \mathrm{sing}(X)} \left( \bigcup_{[L] \in \mathscr{L}_{p,X}} L \right), \end{aligned}$$

and $\bigcup_{[L] \in \mathscr{L}_{p,X}} L$ is the cone $S(p, \mathscr{L}_{p,X}) \subseteq \mathbb{P}^{26}$ over the closed $\mathscr{L}_{p,X} \subseteq \mathscr{L}_{p,\mathbb{P}^{26}} \simeq \mathbb{P}^{25}$ with vertex $p$. Hence $U \subseteq X$ is an open set and moreover $U \neq \emptyset$, by the hypothesis $\mathrm{Vert}(X) = \emptyset$. Thus, applying Proposition 1.18, we get that, for the general point $x \in X$, $\mathrm{sing}(\mathscr{L}_{x,X}) = \emptyset$. Now, let $x \in Y$ be a general point (and hence also a general point of $X$). By Remark A.3, we obtain that $\mathscr{L}_{x,Y} \subset \mathbb{P}^{14}$ is a smooth hyperplane section of $S^{10} \subset \mathbb{P}^{15}$. Since $\mathscr{L}_{x,X}$ is a smooth extension of $\mathscr{L}_{x,Y}$, applying Theorem 2.8, it follows that $S^{10}$ is an irreducible component of $\mathscr{L}_{x,X}$. Now, by Proposition 1.19, it follows that $\mathscr{L}_{x,X}$ is a closed subscheme of $B_{x,X} = \mathrm{Bs}(|II_{x,X}|)$ and hence

$$|II_{x,X}| \subseteq \mathbb{P}(H^0(\mathbb{P}^{15}, \mathscr{I}_{B_{x,X}}(2))) \subseteq \mathbb{P}(H^0(\mathbb{P}^{15}, \mathscr{I}_{S^{10}}(2))) = |II_{x,E_6}|.$$

Moreover, by the commutativity of the diagram

$$
\begin{tikzcd}
E_X \arrow[r, hook] \arrow[rrr, dashed, bend left, "\phi_{|II_{x,X}|}"] & \mathrm{Bl}_x(X) \arrow[r] & X \arrow[dr, dashed, "\tau_{x,X}"] & \\
E_Y \arrow[r, hook] \arrow[u, hook] \arrow[rrr, dashed, bend right, "\phi_{|II_{x,Y}|}"'] & \mathrm{Bl}_x(Y) \arrow[r] \arrow[u, hook] & Y \arrow[u, hook] \arrow[r, dashed, "\tau_{x,Y}"] & \mathbb{P}^9
\end{tikzcd}
$$

and by Proposition 1.16, it follows

$$\mathbb{P}^9 \supseteq \left\langle \overline{\phi_{|II_{x,X}|}(E_X)} \right\rangle \supseteq \left\langle \overline{\phi_{|II_{x,X}|}(E_Y)} \right\rangle = \left\langle \overline{\phi_{|II_{x,Y}|}(E_Y)} \right\rangle = \mathbb{P}^9,$$

from which $\dim(|II_{x,X}|) = 9$ and then $|II_{x,X}| = |II_{x,E_6}|$. Finally, by [Zak93, p. 57, case E], $E_6$ is a Hermitian symmetric space and it follows $X = E_6$ by the main result in [Lan06]. $\square$

Proposition A.4 and Theorem 5.19 are in connection with a fairly well-known open problem, namely to classify all smooth $(L)QEL$-varieties $X$ of type $\delta(X) = (\dim(X) - 1)/2$.

The following result is contained in [Fuj82] and [Ohn97], but it can also be inferred from Theorems 2.14, 2.19, 2.7, 2.8.

**Proposition A.5.** *Let $X \subset \mathbb{P}^n$ be a smooth linearly normal r-dimensional QEL-variety of type $\delta = (r-1)/2 > 0$ with $\mathrm{Sec}(X) \subsetneq \mathbb{P}^n$. Then either $n = 25$, $r = 15$, $\delta = 7$ or $X$ is projectively equivalent to one of the following:*

1. *the Veronese 3-fold $\nu_2(\mathbb{P}^3) \subset \mathbb{P}^9$;*
2. *the blow-up of $\mathbb{P}^3$ at a point $\mathrm{Bl}_p(\mathbb{P}^3) \subset \mathbb{P}^8$;*
3. *a hyperplane section of the Segre 4-fold $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;*
4. *the Segre 5-fold $\mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}$;*
5. *a hyperplane section of the Grassmannian $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$.*

We feel motivated to make the following:

**Conjecture A.6.** Let $X \subset \mathbb{P}^n$ be as in Proposition A.5 with $n = 25$, $r = 15$ and $\delta = 7$. Then $X$ is a hyperplane section of $E_6 \subset \mathbb{P}^{26}$.

# Chapter 6

# On special quadratic birational transformations whose base locus has dimension at most three

In this chapter we continue the study of special quadratic birational transformations $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{S} := \overline{\varphi(\mathbb{P}^n)} \subseteq \mathbb{P}^N$ started in Chapter 5, by reinterpreting techniques and well-known results on special Cremona transformations. While in Chapter 5 we required that $\mathbf{S}$ was a hypersurface, here we allow more freedom in the choice of $\mathbf{S}$, but we only treat the case in which the dimension of the base locus $\mathfrak{B}$ is $r = \dim(\mathfrak{B}) \leq 3$.

Note that for every closed subscheme $X \subset \mathbb{P}^{n-1}$ cut out by the quadrics containing it, we can consider $\mathbb{P}^{n-1}$ as a hyperplane in $\mathbb{P}^n$ and hence $X$ as a subscheme of $\mathbb{P}^n$. So the linear system $|\mathscr{I}_{X,\mathbb{P}^n}(2)|$ of all quadrics in $\mathbb{P}^n$ containing $X$ defines a quadratic rational map $\psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ ($N = h^0(\mathscr{I}_{X,\mathbb{P}^n}(2)) - 1 = n + h^0(\mathscr{I}_{X,\mathbb{P}^{n-1}}(2))$), which is birational onto the image and whose inverse is defined by linear forms, i.e. $\psi$ is of type $(2,1)$. Conversely, every birational transformation $\psi : \mathbb{P}^n \dashrightarrow \overline{\psi(\mathbb{P}^n)} \subseteq \mathbb{P}^N$ of type $(2,1)$ whose image is nondegenerate, normal and linearly normal arise in this way. From this it follows that there are many (special) quadratic transformations. However, when the image $\mathbf{S}$ of the transformation $\varphi$ is sufficiently regular, by straightforward generalization of Proposition 5.2, we obtain strong numerical and geometric restrictions on the base locus $\mathfrak{B}$. For example, as soon as $\mathbf{S}$ is not too much singular, the secant variety $\mathrm{Sec}(\mathfrak{B}) \subset \mathbb{P}^n$ has to be a hypersurface and $\mathfrak{B}$ has to be a *QEL*-variety of type $\delta = \delta(\mathfrak{B}) = 2\dim(\mathfrak{B}) + 2 - n$; in particular $n \leq 2\dim(\mathfrak{B}) + 2$ and $\mathrm{Sec}(\mathfrak{B})$ is a hyperplane if and only if $\varphi$ is of type $(2,1)$. So the classification of transformations $\varphi$ of type $(2,1)$ whose base locus has dimension $\leq 3$ essentially follows from classification results on *QEL*-manifold: Proposition 2.16, Theorem 2.19 and [CMR04, Theorems 4.10 and 7.1].

When $\varphi$ is of type $(2,d)$ with $d \geq 2$, then $\mathrm{Sec}(\mathfrak{B})$ is a nonlinear hypersurface and it is not so easy to exhibit examples. The most difficult cases of this kind are those for which $n = 2r + 2$ i.e. $\delta = 0$. In order to classify these transformations, we proceed as in Propositions 5.21 and 5.23 (see also [MR05]). That is, we first determine the Hilbert polynomial of $\mathfrak{B}$ in Lemmas 6.15 and 6.19, by using the usual Castelnuovo's argument, Castelnuovo's bound and some refinement of Castelnuovo's bound (see Chap. 3); consequently we deduce Propositions 6.17 and 6.24 by applying the classification of smooth varieties of low degree: [Ion84], [Ion86a], [Ion90], [FL94], [FL97], [BB05a], [Ion85]. We also apply the double point formula in Lemmas: 6.16, 6.20, 6.21, 6.22 and 6.23, in order to obtain additional informations on $d$ and $\Delta=\deg(\mathbf{S})$.

We summarize our classification results in Table 6.1. In particular, we provide an answer to a question left open in the recent preprint [AS12].

## 6.1 Notation and general results

Throughout the chapter we work over $\mathbb{C}$ and keep the following setting.

**Assumption 6.1.** Let $\varphi:\mathbb{P}^n\dashrightarrow\mathbf{S}:=\overline{\varphi(\mathbb{P}^n)}\subseteq\mathbb{P}^{n+a}$ be a quadratic birational transformation with smooth connected base locus $\mathfrak{B}$ and with $\mathbf{S}$ nondegenerate, linearly normal and factorial.

Recall that we can resolve the indeterminacies of $\varphi$ with the diagram

$$\begin{tikzcd} & \widetilde{\mathbb{P}^n} \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & \mathbf{S} \end{tikzcd} \tag{6.1}$$

where $\pi:\widetilde{\mathbb{P}^n}=\mathrm{Bl}_{\mathfrak{B}}(\mathbb{P}^n)\to\mathbb{P}^n$ is the blow-up of $\mathbb{P}^n$ along $\mathfrak{B}$ and $\pi'=\varphi\circ\pi:\widetilde{\mathbb{P}^n}\to\mathbf{S}$. Denote by $\mathfrak{B}'$ the base locus of $\varphi^{-1}$, $E$ the exceptional divisor of $\pi$, $E'=\pi'^{-1}(\mathfrak{B}')$, $H=\pi^*(H_{\mathbb{P}^n})$, $H'=\pi'^*(H_{\mathbf{S}})$, and note that, since $\pi'|_{\widetilde{\mathbb{P}^n}\setminus E'}:\widetilde{\mathbb{P}^n}\setminus E'\to\mathbf{S}\setminus\mathfrak{B}'$ is an isomorphism, we have $(\mathrm{sing}(\mathbf{S}))_{\mathrm{red}}\subseteq(\mathfrak{B}')_{\mathrm{red}}$. We also put $r=\dim(\mathfrak{B})$, $r'=\dim(\mathfrak{B}')$, $\lambda=\deg(\mathfrak{B})$, $g=g(\mathfrak{B})$ the sectional genus of $\mathfrak{B}$, $c_j=c_j(\mathscr{T}_{\mathfrak{B}})\cdot H_{\mathfrak{B}}^{r-j}$ (resp. $s_j=s_j(\mathscr{N}_{\mathfrak{B},\mathbb{P}^n})\cdot H_{\mathfrak{B}}^{r-j}$) the degree of the $j$-th Chern class (resp. Segre class) of $\mathfrak{B}$, $\Delta=\deg(\mathbf{S})$, $c=c(\mathbf{S})$ the *coindex* of $\mathbf{S}$ (the last of which is defined by $-K_{\mathrm{reg}(\mathbf{S})}\sim(n+1-c)H_{\mathrm{reg}(\mathbf{S})}$, whenever $\mathrm{Pic}(\mathbf{S})=\mathbb{Z}\langle H_{\mathbf{S}}\rangle$).

**Assumption 6.2.** We suppose that there exists a rational map $\widehat{\varphi}:\mathbb{P}^{n+a}\dashrightarrow\mathbb{P}^n$ defined by a sublinear system of $|\mathscr{O}_{\mathbb{P}^{n+a}}(d)|$ and having base locus $\widehat{\mathfrak{B}}$ such that $\varphi^{-1}=\widehat{\varphi}|_{\mathbf{S}}$ and $\mathfrak{B}'=\widehat{\mathfrak{B}}\cap\mathbf{S}$. We then will say that $\varphi^{-1}$ is *liftable*[1] and that $\varphi$ is *of type* $(2,d)$.

The above assumption yields the relations:

$$\begin{array}{ll} H'\sim 2H-E, & H\sim dH'-E', \\ E'\sim(2d-1)H-dE, & E\sim(2d-1)H'-2E', \end{array} \tag{6.2}$$

and hence also $\mathrm{Pic}(\widetilde{\mathbb{P}^n})\simeq\mathbb{Z}\langle H\rangle\oplus\mathbb{Z}\langle E\rangle\simeq\mathbb{Z}\langle H'\rangle\oplus\mathbb{Z}\langle E'\rangle$. Note that, by the proofs of [ESB89, Proposition 1.3 and 2.1(a)] and by factoriality of $\mathbf{S}$, we obtain that $E'$ is a reduced and irreducible divisor. Moreover we have $\mathrm{Pic}(\mathbf{S})\simeq\mathrm{Pic}(\mathbf{S}\setminus\mathfrak{B}')\simeq\mathrm{Pic}(\widetilde{\mathbb{P}^n}\setminus E')\simeq\mathbb{Z}\langle H'\rangle\simeq\mathbb{Z}\langle H_{\mathbf{S}}\rangle$. Finally, we require the following:

[1] If $a\geq 2$ and $\psi:\mathbb{P}^n\dashrightarrow\mathbf{Z}:=\overline{\psi(\mathbb{P}^n)}\subset\mathbb{P}^{n+a}$ is a birational transformation with $\mathbf{Z}$ factorial, from [MP09] it follows that there exists a Cremona transformation $\widetilde{\psi}:\mathbb{P}^{n+a}\dashrightarrow\mathbb{P}^{n+a}$ such that $\overline{\widetilde{\psi}(\mathbf{Z})}\simeq\mathbb{P}^n\subset\mathbb{P}^{n+a}$ and $\psi^{-1}=\widetilde{\psi}|_{\mathbf{Z}}$; in particular, if $\varpi$ denotes the linear projection of $\mathbb{P}^{n+a}$ onto $\overline{\widetilde{\psi}(\mathbf{Z})}$, we have $\psi^{-1}=(\varpi\circ\widetilde{\psi})|_{\mathbf{Z}}$. But this in general does not ensure the liftability of $\psi^{-1}$, because we only have that $\mathrm{Bs}(\psi^{-1})\subseteq\mathrm{Bs}(\varpi\circ\widetilde{\psi})\cap\mathbf{Z}$.

**Assumption 6.3.** $(\mathrm{sing}(\mathbf{S}))_{\mathrm{red}} \neq (\mathfrak{B}')_{\mathrm{red}}$.

Now we point out that, just as in Proposition 5.2, since $E'$ is irreducible, by Assumption 6.3 and [ESB89, Theorem 1.1], we deduce that $\pi'|_V : V \to U$ coincides with the blow-up of $U$ along $Z$, where $U = \mathrm{reg}(\mathbf{S}) \setminus \mathrm{sing}((\mathfrak{B}')_{\mathrm{red}})$, $V = \pi'^{-1}(U)$ and $Z = U \cap (\mathfrak{B}')_{\mathrm{red}}$. It follows that $K_{\widetilde{\mathbb{P}^n}} \sim (-n-1)H + (n-r-1)E \sim (c-n-1)H' + (n-r'-1)E'$, from which, together with (6.2), we obtain $2r+3-n = n-r'-1$ and $c = (1-2d)r + dn - 3d + 2$. One can also easily see that, for the general point $x \in \mathrm{Sec}(\mathfrak{B}) \setminus \mathfrak{B}$, $\overline{\varphi^{-1}(\varphi(x))}$ is a linear space of dimension $n-r'-1$ and $\overline{\varphi^{-1}(\varphi(x))} \cap \mathfrak{B}$ is a quadric hypersurface, which coincides with the entry locus $\Sigma_x(\mathfrak{B})$ of $\mathfrak{B}$ with respect to $x$. So we can generalize Proposition 5.2, obtaining one of the main results useful for purposes of this chapter:

**Proposition 6.4.** $\mathrm{Sec}(\mathfrak{B}) \subset \mathbb{P}^n$ *is a hypersurface of degree* $2d-1$ *and* $\mathfrak{B}$ *is a QEL-variety of type* $\delta = 2r+2-n$.

In many cases, $\mathfrak{B}$ has a much stronger property of being *QEL*-variety. Recall that a subscheme $X \subset \mathbb{P}^n$ is said to have the $K_2$ property if $X$ is cut out by quadratic forms $F_0, \ldots, F_N$ such that the Koszul relations among the $F_i$ are generated by linear syzygies. We have the following fact (see [Ver01] and [Alz08]):

**Fact 6.5.** *Let* $X \subset \mathbb{P}^n$ *be a smooth variety cut out by quadratic forms* $F_0, \ldots, F_N$ *satisfying* $K_2$ *property and let* $F = [F_0, \ldots, F_N] : \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ *be the induced rational map. Then for every* $x \in \mathbb{P}^n \setminus X$, $\overline{F^{-1}(F(x))}$ *is a linear space of dimension* $n+1-\mathrm{rank}\left((\partial F_i/\partial x_j(x))_{i,j}\right)$*; moreover,* $\dim(\overline{F^{-1}(F(x))}) > 0$ *if and only if* $x \in \mathrm{Sec}(X) \setminus X$ *and in this case* $\overline{F^{-1}(F(x))} \cap X$ *is a quadric hypersurface, which coincides with the entry locus* $\Sigma_x(X)$ *of* $X$ *with respect to* $x$.

We have a simple sufficient condition for the $K_2$ property (see [SD72], [GL88] and [AR03, Proposition 2]):

**Fact 6.6.** *Let* $X \subset \mathbb{P}^n$ *be a smooth linearly normal variety and suppose* $h^1(\mathscr{O}_X) = 0$ *if* $\dim(X) \geq 2$. *Putting* $\lambda = \deg(X)$ *and* $s = \mathrm{codim}_{\mathbb{P}^n}(X)$ *we have:*

- *if* $\lambda \leq 2s+1$, *then* $X$ *is arithmetically Cohen-Macaulay;*
- *if* $\lambda \leq 2s$, *then the homogeneous ideal of* $X$ *is generated by quadratic forms;*
- *if* $\lambda \leq 2s-1$, *then the syzygies of the generators of the homogeneous ideal of* $X$ *are generated by the linear ones.*

*Remark* 6.7. Let $\psi : \mathbb{P}^n \dashrightarrow \mathbf{Z} := \overline{\psi(\mathbb{P}^n)} \subseteq \mathbb{P}^{n+a}$ be a birational transformation ($n \geq 3$).

We point out that, from Grothendieck's Theorem on parafactoriality (Samuel's Conjecture) [Gro68, XI Corollaire 3.14] it follows that $\mathbf{Z}$ is factorial whenever it is a local complete intersection with $\dim(\mathrm{sing}(\mathbf{Z})) < \dim(\mathbf{Z}) - 3$. Of course, every complete intersection in a smooth variety is a local complete intersection.

Moreover, $\psi^{-1}$ is liftable whenever $\mathrm{Pic}(\mathbf{Z}) = \mathbb{Z}\langle H_{\mathbf{Z}} \rangle$ and $\mathbf{Z}$ is factorial and projectively normal. So, from [Lar73] and [Har70, IV Corollary 3.2], $\psi^{-1}$ is liftable whenever $\mathbf{Z}$ is either smooth and projectively normal with $n \geq a+2$ or a factorial complete intersection.

## 6.2 Numerical restrictions

Proposition 6.4 already provides a restriction on the invariants of the transformation $\varphi$; here we give further restrictions of this kind.

**Proposition 6.8.** *Let $\varepsilon = 0$ if $\langle \mathfrak{B} \rangle = \mathbb{P}^n$ and let $\varepsilon = 1$ otherwise.*

- *If $r = 1$ we have:*

$$\begin{aligned} \lambda &= \frac{n^2 - n + 2\varepsilon - 2a - 2}{2}, \\ g &= \frac{n^2 - 3n + 4\varepsilon - 2a - 2}{2}. \end{aligned}$$

- *If $r = 2$ we have:*

$$\begin{aligned} \chi(\mathscr{O}_{\mathfrak{B}}) &= \frac{2a - n^2 + 5n + 2g - 6\varepsilon + 4}{4}, \\ \lambda &= \frac{n^2 - n + 2g + 2\varepsilon - 2a - 4}{4}. \end{aligned}$$

- *If $r = 3$ we have:*

$$\chi(\mathscr{O}_{\mathfrak{B}}) = \frac{4\lambda - n^2 + 3n - 2g - 4\varepsilon + 2a + 6}{2}.$$

*Proof.* By Proposition 6.4 we have $h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(1)) = \varepsilon$. Since $\mathbf{S}$ is normal and linearly normal, we have $h^0(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(2)) = n + 1 + a$ (see Lemma 4.2). Moreover, since $n \leq 2r + 2$ (being $\delta \geq 0$), proceeding as in Lemma 5.4 (or applying [MR05, Proposition 1.8]), we obtain $h^j(\mathbb{P}^n, \mathscr{I}_{\mathfrak{B}}(k)) = 0$ for every $j, k \geq 1$. So we obtain $\chi(\mathscr{O}_{\mathfrak{B}}(1)) = n + 1 - \varepsilon$ and $\chi(\mathscr{O}_{\mathfrak{B}}(2)) = (n+1)(n+2)/2 - (n + 1 + a)$. □

**Proposition 6.9.**

- *If $r = 1$ we have:*

$$\begin{aligned} c_1 &= 2 - 2g, \\ s_1 &= -n\lambda - \lambda - 2g + 2, \\ d &= \frac{2\lambda - 2^n}{2n\lambda - 2\lambda - 2^{n+1} - 4g + 4}, \\ \Delta &= -n\lambda + \lambda + 2^n + 2g - 2. \end{aligned}$$

- *If* $r = 2$ *we have:*

$$\begin{aligned} c_1 &= \lambda - 2g + 2, \\ c_2 &= -\frac{n^2\lambda}{2} + \frac{3n\lambda}{2} + 2^n + 2gn - 2n - 2g - \Delta + 2, \\ s_1 &= -n\lambda - 2g + 2, \\ s_2 &= 2n\lambda + 2^n + 4gn - 4n - \Delta, \\ d\Delta &= -n\lambda + 2\lambda + 2^{n-1} + 2g - 2. \end{aligned}$$

- *If* $r = 3$ *we have:*

$$\begin{aligned} c_1 &= 2\lambda - 2g + 2, \\ c_2 &= -\frac{n^2\lambda}{2} + \frac{5n\lambda}{2} - \lambda + 2^{n-1} + 2gn - 2n - 6g - d\Delta + 6, \\ c_3 &= \frac{n^3\lambda}{3} - 2n^2\lambda + \frac{11n\lambda}{3} - 2\lambda - n2^{n-1} + 3\,2^{n-1} - gn^2 + n^2 + 3gn + \\ &\quad + d\Delta n - 3n - 4g - d\Delta - \Delta + 4, \\ s_1 &= -n\lambda + \lambda - 2g + 2, \\ s_2 &= 2n\lambda - 2\lambda + 2^{n-1} + 4gn - 4n - 4g - d\Delta + 4, \\ s_3 &= \frac{2n^3\lambda}{3} - 4n^2\lambda + \frac{10n\lambda}{3} - n2^n + 2^n - 4gn^2 + 4n^2 + 4gn + 2d\Delta n - 4n - \Delta. \end{aligned}$$

*Proof.* See also [CK89] and [CK91]. By [CK89, page 291] we see that

$$H^j \cdot E^{n-j} = \begin{cases} 1, & \text{if } j = n; \\ 0, & \text{if } r+1 \le j \le n-1; \\ (-1)^{n-j-1} s_{r-j}, & \text{if } j \le r. \end{cases}$$

Since $H' = 2H - E$ and $H = dH' - E'$ we have

$$\begin{aligned} \Delta &= H'^n = (2H - E)^n, && (6.3) \\ d\Delta &= dH'^n = H'^{n-1} \cdot (dH' - E') = (2H - E)^{n-1} \cdot H. && (6.4) \end{aligned}$$

From the exact sequence $0 \to \mathscr{T}_{\mathfrak{B}} \to \mathscr{T}_{\mathbb{P}^n}|_{\mathfrak{B}} \to \mathscr{N}_{\mathfrak{B},\mathbb{P}^n} \to 0$ we get:

$$\begin{aligned} s_1 &= -\lambda(n+1) + c_1, && (6.5) \\ s_2 &= \lambda\binom{n+2}{2} - c_1(n+1) + c_2, && (6.6) \\ s_3 &= -\lambda\binom{n+3}{3} + c_1\binom{n+2}{2} - c_2(n+1) + c_3, && (6.7) \\ &\vdots \end{aligned}$$

Moreover $c_1 = -K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^{r-1}$ and it can be expressed as a function of $\lambda$ and $g$. Thus we found $r+3$ independent equations on the $2r+5$ variables: $c_1, \ldots, c_r, s_1, \ldots, s_r, d, \Delta, \lambda, g, n$. □

*Remark* 6.10. Proposition 6.9 holds under less restrictive assumptions, as shown in the above proof. Here we treat the special case: let $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{Z} := \overline{\psi(\mathbb{P}^8)} \subseteq \mathbb{P}^{8+a}$ be a quadratic rational map whose base locus is a smooth irreducible 3-dimensional variety $X$. Without any other restriction on $\psi$, denoting with $\pi : \mathrm{Bl}_X(\mathbb{P}^8) \to \mathbb{P}^8$ the blow-up of $\mathbb{P}^8$ along $X$ and with $s_i(X) = s_i(\mathscr{N}_{X,\mathbb{P}^8})$, we have

$$\deg(\psi)\deg(\mathbf{Z}) = (2\pi^*(H_{\mathbb{P}^8}) - E_X)^8 = -s_3(X) - 16s_2(X) - 112s_1(X) - 448\deg(X) + 256. \tag{6.8}$$

Moreover, if $\psi$ is birational with liftable inverse and $\dim(\mathrm{sing}(\mathbf{Z})) \leq 6$, we also have

$$d\deg(\mathbf{Z}) = (2\pi^*(H_{\mathbb{P}^8}) - E_X)^7 \cdot \pi^*(H_{\mathbb{P}^8}) = -s_2(X) - 14s_1(X) - 84\deg(X) + 128, \tag{6.9}$$

where $d$ denotes the degree of the linear system defining $\psi^{-1}$.

Proposition 6.11 is a translation of the well-known *double point formula* (see for example [PS76] and [Lak76]), taking into account Proposition 6.4.

**Proposition 6.11.** *If $\delta = 0$ then*

$$2(2d-1) = \lambda^2 - \sum_{j=0}^{r} \binom{2r+1}{j} s_{r-j}(\mathscr{T}_{\mathfrak{B}}) \cdot H_{\mathfrak{B}}^j.$$

## 6.3 Case of dimension 1

Lemma 6.12 directly follows from Propositions 6.8 and 6.9.

**Lemma 6.12.** *If $r = 1$, then one of the following cases holds:*

*(A)* $n = 3$, $a = 1$, $\lambda = 2$, $g = 0$, $d = 1$, $\Delta = 2$;

*(B)* $n = 4$, $a = 0$, $\lambda = 5$, $g = 1$, $d = 3$, $\Delta = 1$;

*(C)* $n = 4$, $a = 1$, $\lambda = 4$, $g = 0$, $d = 2$, $\Delta = 2$;

*(D)* $n = 4$, $a = 2$, $\lambda = 4$, $g = 1$, $d = 1$, $\Delta = 4$;

*(E)* $n = 4$, $a = 3$, $\lambda = 3$, $g = 0$, $d = 1$, $\Delta = 5$.

**Proposition 6.13.** *If $r = 1$, then one of the following cases holds:*

*(I)* $n = 3$, $a = 1$, $\mathfrak{B}$ *is a conic;*

*(II)* $n = 4$, $a = 0$, $\mathfrak{B}$ *is an elliptic curve of degree* 5*;*

*(III)* $n = 4$, $a = 1$, $\mathfrak{B}$ *is the rational normal quartic curve;*

*(IV)* $n = 4$, $a = 3$, $\mathfrak{B}$ *is the twisted cubic curve.*

*Proof.* From Lemma 6.12 it remains only to exclude case (D). In this case $\mathfrak{B}$ is a complete intersection of two quadrics in $\mathbb{P}^3$ and also it is an *OADP*-curve. This is absurd because the only *OADP*-curve is the twisted cubic curve. □

## 6.4 Case of dimension 2

Proposition 6.14 follows from Proposition 2.16 and [CMR04, Theorem 4.10].

**Proposition 6.14.** *If $r=2$, then either $n=6$, $d\geq 2$, $\langle\mathfrak{B}\rangle=\mathbb{P}^6$, or one of the following cases holds:*

*(V)* $n=4$, $d=1$, $\delta=2$, $\mathfrak{B}=\mathbb{P}^1\times\mathbb{P}^1\subset\mathbb{P}^3\subset\mathbb{P}^4$;

*(VI)* $n=5$, $d=1$, $\delta=1$, $\mathfrak{B}$ *is a hyperplane section of* $\mathbb{P}^1\times\mathbb{P}^2\subset\mathbb{P}^5$;

*(VII)* $n=5$, $d=2$, $\delta=1$, $\mathfrak{B}=\nu_2(\mathbb{P}^2)\subset\mathbb{P}^5$ *is the Veronese surface;*

*(VIII)* $n=6$, $d=1$, $\delta=0$, $\mathfrak{B}\subset\mathbb{P}^5$ *is an OADP-surface, i.e.* $\mathfrak{B}$ *is as in one of the following cases:*

- *$(VIII_1)$* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(3))$ *or* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2)\oplus\mathscr{O}(2))$;
- *$(VIII_2)$ del Pezzo surface of degree 5 (hence the blow-up of* $\mathbb{P}^2$ *at 4 points* $p_1,\dots,p_4$ *and* $|H_{\mathfrak{B}}|=|3H_{\mathbb{P}^2}-p_1-\cdots-p_4|$*).*

**Lemma 6.15.** *If $r=2$, $n=6$ and $\langle\mathfrak{B}\rangle=\mathbb{P}^6$, then one of the following cases holds:*

*(A)* $a=0$, $\lambda=7$, $g=1$, $\chi(\mathscr{O}_{\mathfrak{B}})=0$;

*(B)* $0\leq a\leq 3$, $\lambda=8-a$, $g=3-a$, $\chi(\mathscr{O}_{\mathfrak{B}})=1$.

*Proof.* By Proposition 6.8 it follows that $g=2\lambda+a-13$ and $\chi(\mathscr{O}_{\mathfrak{B}})=\lambda+a-7$. By Lemma 3.2 and using that $g\geq 0$ (proceeding as in Proposition 5.21), we obtain $(13-a)/2\leq\lambda\leq 8-a$. □

**Lemma 6.16.** *If $r=2$, $n=6$ and $\langle\mathfrak{B}\rangle=\mathbb{P}^6$, then one of the following cases holds:*

- $a=0$, $d=4$, $\Delta=1$;
- $a=1$, $d=3$, $\Delta=2$;
- $a=2$, $d=2$, $\Delta=4$;
- $a=3$, $d=2$, $\Delta=5$.

*Proof.* We have $s_1(\mathscr{T}_{\mathfrak{B}})\cdot H_{\mathfrak{B}}=-c_1$ and $s_2(\mathscr{T}_{\mathfrak{B}})=c_1^2-c_2=12\chi(\mathscr{O}_{\mathfrak{B}})-2c_2$. So, by Proposition 6.11, we obtain

$$2(2d-1)=\lambda^2-10\lambda-12\chi(\mathscr{O}_{\mathfrak{B}})+2c_2+5c_1. \tag{6.10}$$

Now, by Propositions 6.8 and 6.9, we obtain

$$d\Delta=2a+4,\quad \Delta=(g^2+(-2a-4)g-16d+a^2-4a+75)/8, \tag{6.11}$$

and then we conclude by Lemma 6.15. □

**Proposition 6.17.** *If $r=2$, $n=6$ and $\langle\mathfrak{B}\rangle=\mathbb{P}^6$ then one of the following cases holds:*

*(IX)* $a=0$, $\lambda=7$, $g=1$, $\mathfrak{B}$ *is an elliptic scroll* $\mathbb{P}_C(\mathscr{E})$ *with* $e(\mathscr{E})=-1$*;*

*(X)* $a=0$, $\lambda=8$, $g=3$, $\mathfrak{B}$ *is the blow-up of* $\mathbb{P}^2$ *at* $8$ *points* $p_1\ldots,p_8$, $|H_{\mathfrak{B}}|=|4H_{\mathbb{P}^2}-p_1-\cdots-p_8|$*;*

*(XI)* $a=1$, $\lambda=7$, $g=2$, $\mathfrak{B}$ *is the blow-up of* $\mathbb{P}^2$ *at* $6$ *points* $p_0\ldots,p_5$, $|H_{\mathfrak{B}}|=|4H_{\mathbb{P}^2}-2p_0-p_1-\cdots-p_5|$*;*

*(XII)* $a=2$, $\lambda=6$, $g=1$, $\mathfrak{B}$ *is the blow-up of* $\mathbb{P}^2$ *at* $3$ *points* $p_1,p_2,p_3$, $|H_{\mathfrak{B}}|=|3H_{\mathbb{P}^2}-p_1-p_2-p_3|$*;*

*(XIII)* $a=3$, $\lambda=5$, $g=0$, $\mathfrak{B}$ *is a rational normal scroll.*

*Proof.* For $a=0$, $a=1$ and $a\in\{2,3\}$ the statement follows, respectively, from [CK89], Proposition 5.21 and [Ion84]. □

## 6.5 Case of dimension 3

Proposition 6.18 follows from: Proposition 2.16, [Fuj82], Theorems 2.19 and 2.7 and [CMR04].

**Proposition 6.18.** *If* $r=3$*, then either* $n=8$, $d\geq 2$, $\langle\mathfrak{B}\rangle=\mathbb{P}^8$*, or one of the following cases holds:*

*(XIV)* $n=5$, $d=1$, $\delta=3$, $\mathfrak{B}=Q^3\subset\mathbb{P}^4\subset\mathbb{P}^5$ *is a quadric;*

*(XV)* $n=6$, $d=1$, $\delta=2$, $\mathfrak{B}=\mathbb{P}^1\times\mathbb{P}^2\subset\mathbb{P}^5\subset\mathbb{P}^6$*;*

*(XVI)* $n=7$, $d=1$, $\delta=1$, $\mathfrak{B}\subset\mathbb{P}^6$ *is as in one of the following cases:*

- *(XVI₁)* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(1)\oplus\mathscr{O}(2))$*;*
- *(XVI₂) linear section of* $\mathbb{G}(1,4)\subset\mathbb{P}^9$*;*

*(XVII)* $n=7$, $d=2$, $\delta=1$, $\mathfrak{B}$ *is a hyperplane section of* $\mathbb{P}^2\times\mathbb{P}^2\subset\mathbb{P}^8$*;*

*(XVIII)* $n=8$, $d=1$, $\delta=0$, $\mathfrak{B}\subset\mathbb{P}^7$ *is an OADP-variety, i.e.* $\mathfrak{B}$ *is as in one of the following cases:*

- *(XVIII₁)* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(1)\oplus\mathscr{O}(3))$ *or* $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(2)\oplus\mathscr{O}(2))$*;*
- *(XVIII₂) Edge variety of degree* $6$ *(i.e.* $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$*) or Edge variety of degree* $7$*;*
- *(XVIII₃)* $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$*, where* $\mathscr{E}$ *is a vector bundle with* $c_1(\mathscr{E})=4$ *and* $c_2(\mathscr{E})=8$*, given as an extension by the following exact sequence* $0\to\mathscr{O}_{\mathbb{P}^2}\to\mathscr{E}\to\mathscr{I}_{\{p_1,\ldots,p_8\},\mathbb{P}^2}(4)\to 0$.

In the following we denote by $\Lambda\subsetneq C\subsetneq S\subsetneq\mathfrak{B}$ a sequence of general linear sections of $\mathfrak{B}$.

**Lemma 6.19.** *If* $r=3$, $n=8$ *and* $\langle\mathfrak{B}\rangle=\mathbb{P}^8$*, then one of the following cases holds:*

*(A)* $a=0$, $\lambda=13$, $g=8$, $K_S\cdot H_S=1$, $K_S^2=-1$*;*

*(B)* $a = 1$*,* $\lambda = 12$*,* $g = 7$*,* $K_S \cdot H_S = 0$*,* $K_S^2 = 0$*;*

*(C)* $0 \leq a \leq 6$*,* $\lambda = 12 - a$*,* $g = 6 - a$*,* $K_S \cdot H_S = -2 - a$*.*

*Proof.* Firstly we note that, from the exact sequence $0 \to \mathscr{T}_S \to \mathscr{T}_{\mathfrak{B}}|_S \to \mathscr{O}_S(1) \to 0$, we deduce $c_2 = c_2(S) + c_1(S) = 12\chi(\mathscr{O}_S) - K_S^2 - K_S \cdot H_S$ and hence

$$K_S^2 = 14\lambda + 12\chi(\mathscr{O}_S) - 12g + d\Delta - 116 = -22\lambda + 12g + d\Delta - 12a + 184. \tag{6.12}$$

Secondly we note that (see Lemma 3.2), putting $h_\Lambda(2) := h^0(\mathbb{P}^5, \mathscr{O}(2)) - h^0(\mathbb{P}^5, \mathscr{I}_\Lambda(2))$, we have

$$\min\{\lambda, 11\} \leq h_\Lambda(2) \leq 21 - h^0(\mathbb{P}^8, \mathscr{I}_{\mathfrak{B}}(2)) = 12 - a. \tag{6.13}$$

Now we establish the following:

*Claim* 6.19.1. If $K_S \cdot H_S \leq 0$ and $K_S \nsim 0$, then $\lambda = 12 - a$ and $g = 6 - a$.

*Proof of the Claim.* Similarly to Case 5.23.1, we obtain that $P_{\mathfrak{B}}(-1) = 0$ and $P_{\mathfrak{B}}(0) = 1 - q$, where $q := h^1(S, \mathscr{O}_S) = h^1(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}})$; in particular $g = -5q - a + 6$ and $\lambda = -3q - a + 12$. Since $g \geq 0$ we have $5q \leq 6 - a$ and the possibilities are: if $a \leq 1$ then $q \leq 1$; if $a \geq 2$ then $q = 0$. If $(a, q) = (0, 1)$ then $(g, \lambda) = (1, 9)$ and the case is excluded by [Fuj90, Theorem 12.3][2]; if $(a, q) = (1, 1)$ then $(g, \lambda) = (0, 8)$ and the case is excluded by [Fuj90, Theorem 12.1]. Thus we have $q = 0$ and hence $g = 6 - a$ and $\lambda = 12 - a$; in particular we have $a \leq 6$. □

Now we discuss the cases according to the value of $a$.

*Case* 6.19.1 ($a = 0$). It is clear that $\varphi$ must be of type $(2,5)$ and hence $K_S^2 = -22\lambda + 12g + 189$. By Claim 6.19.1, if $K_S \cdot H_S = 2g - 2 - \lambda < 0$, we fall into case (C). So we suppose that $K_S \cdot H_S \geq 0$, namely that $g \geq \lambda/2 + 1$. From Castelnuovo's bound it follows that $\lambda \geq 12$ and if $\lambda = 12$ then $K_S \cdot H_S = 0$, $g = 7$ and hence $K_S^2 = 9$. Since this is impossible by Claim 6.19.1, we conclude that $\lambda \geq 13$. Now by (6.13) it follows that $11 \leq h_\Lambda(2) \leq 12$, but if $h_\Lambda(2) = 11$ from Castelnuovo Lemma (Proposition 3.3) we obtain a contradiction. Thus we have $h_\Lambda(2) = 12$ and $h^0(\mathbb{P}^5, \mathscr{I}_\Lambda(2)) = h^0(\mathbb{P}^8, \mathscr{I}_{\mathfrak{B}}(2)) = 9$. So from Theorem 3.9 we deduce that $\lambda \leq 14$ and furthermore, by the refinement of Castelnuovo's bound contained in Theorem 3.5, we obtain $g \leq 2\lambda - 18$. In summary we have the following possibilities:

(i) $\lambda = 13$, $g = 8$, $K_S \cdot H_S = 1$, $\chi(\mathscr{O}_S) = 2$, $K_S^2 = -1$;

(ii) $\lambda = 14$, $g = 8$, $K_S \cdot H_S = 0$, $\chi(\mathscr{O}_S) = -1$, $K_S^2 = -23$;

(iii) $\lambda = 14$, $g = 9$, $K_S \cdot H_S = 2$, $\chi(\mathscr{O}_S) = 1$, $K_S^2 = -11$;

(iv) $\lambda = 14$, $g = 10$, $K_S \cdot H_S = 4$, $\chi(\mathscr{O}_S) = 3$, $K_S^2 = 1$.

---

[2] Note that $\mathfrak{B}$ cannot be a scroll over a curve (this follows from (6.19) and (6.20) below and also it follows from [MR05, Proposition 3.2(i)]).

Case (i) coincides with case (A). Case (ii) is excluded by Claim 6.19.1. In the circumstances of case (iii), we have $h^1(S,\mathcal{O}_S)=h^2(S,\mathcal{O}_S)=h^0(S,K_S)$. If $h^1(S,\mathcal{O}_S)>0$, since $(K_{\mathfrak{B}}+4H_{\mathfrak{B}})\cdot K_S = K_S^2+3K_S\cdot H_S=-5<0$, we see that $K_{\mathfrak{B}}+4H_{\mathfrak{B}}$ is not nef and then we obtain a contradiction by [Ion86b]. If $h^1(S,\mathcal{O}_S)=0$, then we also have $h^1(\mathfrak{B},\mathcal{O}_{\mathfrak{B}})=h^2(\mathfrak{B},\mathcal{O}_{\mathfrak{B}})=0$ and hence $\chi(\mathcal{O}_{\mathfrak{B}})=1-h^3(\mathfrak{B},\mathcal{O}_{\mathfrak{B}})\leq 1$, against the fact that $\chi(\mathcal{O}_{\mathfrak{B}})=2\lambda-g-17=2$. Thus case (iii) does not occur. Finally, in the circumstances of case (iv), note that $h^0(S,K_S)=2+h^1(S,\mathcal{O}_S)\geq 2$ and we write $|K_S|=|M|+F$, where $|M|$ is the mobile part of the linear system $|K_S|$ and $F$ is the fixed part. If $M_1=M$ is a general member of $|M|$, there exists $M_2\in|M|$ having no common irreducible components with $M_1$ and so $M^2=M_1\cdot M_2=\sum_p (M_1\cdot M_2)_p\geq 0$; furthermore, by using Bertini Theorem, we see that $\mathrm{sing}(M_1)$ consists of points $p$ such that the intersection multiplicity $(M_1\cdot M_2)_p$ of $M_1$ and $M_2$ in $p$ is at least 2. By definition, we also have $M\cdot F\geq 0$ and so we deduce $2p_a(M)-2=M\cdot(M+K_S)=2M^2+M\cdot F\geq 0$, from which $p_a(M)\geq 1$ and $p_a(M)=2$ if $F=0$. On the other hand, we have $M\cdot H_S\leq K_S\cdot H_S=4$ and, since $S$ is cut out by quadrics, $M$ does not contain planar curves of degree $\geq 3$. If $M\cdot H_S=4$, then $F=0$, $M^2=1$ and $M$ is a (possibly disconnected) smooth curve; since $p_a(M)=2$, $M$ is actually disconnected and so it is a disjoint union of twisted cubics, conics and lines. But then we obtain the contradiction that $p_a(M)=1-\#\{\text{connected components of } M\}<0$. If $M\cdot H_S\leq 3$, then $M$ must be either a twisted cubic or a union of conics and lines. In all these cases we again obtain the contradiction that $p_a(M)=1-\#\{\text{connected components of } M\}\leq 0$. Thus case (iv) does not occur.

*Case* 6.19.2 ($a=1$). By Proposition 5.23 we fall into case (B) or (C).

*Case* 6.19.3 ($a\geq 2$). By (6.13) it follows that $\lambda\leq 10$ and by Castelnuovo's bound it follows that $K_S\cdot H_S\leq -4<0$. Thus, by Claim 6.19.1 we fall into case (C).

□

Now we apply the double point formula (Proposition 6.11) in order to obtain additional numerical restrictions under the hypothesis of Lemma 6.19.

**Lemma 6.20.** *If $r=3$, $n=8$ and $\langle\mathfrak{B}\rangle=\mathbb{P}^8$, then*

$$K_{\mathfrak{B}}^3=\lambda^2+23\lambda-24g-(7d+1)\Delta-4d+36a-226.$$

*Proof.* We have (see [Har77, App. A, Exercise 6.7]):

$$\begin{aligned}
s_1(\mathscr{T}_{\mathfrak{B}})\cdot H_{\mathfrak{B}}^2 &= -c_1(\mathfrak{B})\cdot H_{\mathfrak{B}}^2=K_{\mathfrak{B}}\cdot H_{\mathfrak{B}}^2,\\
s_2(\mathscr{T}_{\mathfrak{B}})\cdot H_{\mathfrak{B}} &= c_1(\mathfrak{B})^2\cdot H_{\mathfrak{B}}-c_2(\mathfrak{B})\cdot H_{\mathfrak{B}}=K_{\mathfrak{B}}^2\cdot H_{\mathfrak{B}}-c_2(\mathfrak{B})\cdot H_{\mathfrak{B}}\\
&= 3K_{\mathfrak{B}}\cdot H_{\mathfrak{B}}^2-2H_{\mathfrak{B}}^3-2c_2(\mathfrak{B})\cdot H_{\mathfrak{B}}+12\left(\chi(\mathcal{O}_{\mathfrak{B}}(H_{\mathfrak{B}}))-\chi(\mathcal{O}_{\mathfrak{B}})\right),\\
s_3(\mathscr{T}_{\mathfrak{B}}) &= -c_1(\mathfrak{B})^3+2c_1(\mathfrak{B})\cdot c_2(\mathfrak{B})-c_3(\mathfrak{B})=K_{\mathfrak{B}}^3+48\chi(\mathcal{O}_{\mathfrak{B}})-c_3(\mathfrak{B}).
\end{aligned}$$

Hence, applying the double point formula and using the relations $\chi(\mathcal{O}_{\mathfrak{B}})=2\lambda-g+a-17$,

$\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}}))=9$, we obtain:

$$\begin{aligned}
4d-2 &= 2\deg(\mathrm{Sec}(\mathfrak{B}))\\
&= \deg(\mathfrak{B})^2 - s_3(\mathscr{T}_{\mathfrak{B}}) - 7\,s_2(\mathscr{T}_{\mathfrak{B}})\cdot H_{\mathfrak{B}} - 21\,s_1(\mathscr{T}_{\mathfrak{B}})\cdot H_{\mathfrak{B}}^2 - 35\,H_{\mathfrak{B}}^3\\
&= \deg(\mathfrak{B})^2 - 21\deg(\mathfrak{B}) - 42\,K_{\mathfrak{B}}\cdot H_{\mathfrak{B}}^2 + 14\,c_2(\mathfrak{B})\cdot H_{\mathfrak{B}} - K_{\mathfrak{B}}^3\\
&\quad + c_3(\mathfrak{B}) - 84\,\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) + 36\,\chi(\mathscr{O}_{\mathfrak{B}})\\
&= -K_{\mathfrak{B}}^3 + \lambda^2 + 23\lambda - 24g - (7d+1)\Delta + 36a - 228.
\end{aligned}$$

□

**Lemma 6.21.** *If $r=3$, $n=8$, $\langle\mathfrak{B}\rangle=\mathbb{P}^8$ and $\mathfrak{B}$ is a quadric fibration over a curve, then one of the following cases holds:*

- $a=3$, $\lambda=9$, $g=3$, $d=3$, $\Delta=5$;
- $a=4$, $\lambda=8$, $g=2$, $d=2$, $\Delta=10$.

*Proof.* Denote by $\beta:(\mathfrak{B},H_{\mathfrak{B}})\to(Y,H_Y)$ the projection over the curve $Y$ such that $\beta^*(H_Y)=K_{\mathfrak{B}}+2H_{\mathfrak{B}}$. We have

$$\begin{aligned}
0 &= \beta^*(H_Y)^2\cdot H_{\mathfrak{B}} = K_{\mathfrak{B}}^2\cdot H_{\mathfrak{B}} + 4K_{\mathfrak{B}}\cdot H_{\mathfrak{B}}^2 + 4H_{\mathfrak{B}}^3,\\
0 &= \beta^*(H_Y)^3 = K_{\mathfrak{B}}^3 + 6K_{\mathfrak{B}}^2\cdot H_{\mathfrak{B}} + 12K_{\mathfrak{B}}\cdot H_{\mathfrak{B}}^2 + 8H_{\mathfrak{B}}^3,\\
\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) &= \frac{1}{12}K_{\mathfrak{B}}^2\cdot H_{\mathfrak{B}} - \frac{1}{4}K_{\mathfrak{B}}\cdot H_{\mathfrak{B}}^2 + \frac{1}{6}H_{\mathfrak{B}}^3 + \frac{1}{12}c_2(\mathfrak{B})\cdot H_{\mathfrak{B}} + \chi(\mathscr{O}_{\mathfrak{B}}),
\end{aligned}$$

from which it follows that

$$K_{\mathfrak{B}}^3 = -8\lambda + 24g - 24, \tag{6.14}$$
$$c_2(\mathfrak{B})\cdot H_{\mathfrak{B}} = -36\lambda + 26g - 12a + 298. \tag{6.15}$$

Hence, by Lemma 6.20 and Proposition 6.9, we obtain

$$d\Delta = 23\lambda - 16g + 12a - 180, \tag{6.16}$$
$$\Delta + 4d = \lambda^2 - 130\lambda + 64g - 48a + 1058. \tag{6.17}$$

Now the conclusion follows from Lemma 6.19, by observing that the case $a=6$ cannot occur. In fact, if $a=6$, by [Ion84] it follows that $\mathfrak{B}$ is a rational normal scroll and by a direct calculation (or by Lemma 6.23) we see that $d=2$ and $\Delta=14$. □

**Lemma 6.22.** *If $r=3$, $n=8$, $\langle\mathfrak{B}\rangle=\mathbb{P}^8$ and $\mathfrak{B}$ is a scroll over a smooth surface $Y$, then we have:*

$$\begin{aligned}
c_2(Y) &= \big((7d-1)\,\lambda^2 + (177-679d)\,\lambda + (292d-92)\,g - 28d^2\\
&\quad + (5554-252a)\,d + 36a - 1474\big)\,/\,(2d+2)\,,\\
\Delta &= \big(\lambda^2 - 107\lambda + 48g - 4d - 36a + 878\big)\,/\,(d+1)\,.
\end{aligned}$$

*Proof.* Similarly to Lemma 6.21, denote by $\beta : (\mathfrak{B}, H_{\mathfrak{B}}) \to (Y, H_Y)$ the projection over the surface $Y$ such that $\beta^*(H_Y) = K_{\mathfrak{B}} + 2H_{\mathfrak{B}}$. Since $\beta^*(H_Y)^3 = 0$ we obtain

$$\begin{aligned} K_{\mathfrak{B}}^3 &= -8H_{\mathfrak{B}}^3 - 12K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 - 6K_{\mathfrak{B}}^2 \cdot H_{\mathfrak{B}} \\ &= -30K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^2 + 4H_{\mathfrak{B}}^3 + 6c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}} - 72\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) + 72\chi(\mathscr{O}_{\mathfrak{B}}) \\ &= 130\lambda - 72g - 6d\Delta + 72a - 1104. \end{aligned}$$

Now we conclude comparing the last formula with Lemma 6.20 and using the relation

$$70\lambda - 44g + (7d - 1)\Delta - 596 = c_3(\mathfrak{B}) = c_1(\mathbb{P}^1)c_2(Y) = 2c_2(Y). \tag{6.18}$$

□

**Lemma 6.23.** *If $r = 3$, $n = 8$, $\langle\mathfrak{B}\rangle = \mathbb{P}^8$ and $\mathfrak{B}$ is a scroll over a smooth curve, then we have: $a = 6$, $\lambda = 6$, $g = 0$, $d = 2$, $\Delta = 14$.*

*Proof.* We have a projection $\beta : (\mathfrak{B}, H_{\mathfrak{B}}) \to (Y, H_Y)$ over a curve $Y$ such that $\beta^*(H_Y) = K_{\mathfrak{B}} + 3H_{\mathfrak{B}}$. By expanding the expressions $\beta^*(H_Y)^2 \cdot H_{\mathfrak{B}} = 0$ and $\beta^*(H_Y)^3 = 0$ we obtain $K_{\mathfrak{B}}^2 \cdot H_{\mathfrak{B}} = 3\lambda - 12g + 12$ and $K_{\mathfrak{B}}^3 = 54(g-1)$, and hence by Lemma 6.20 we get

$$\lambda^2 + 23\lambda - 78g - (7d + 1)\Delta - 4d + 36a - 172 = 0. \tag{6.19}$$

Also, by expanding the expression $\chi(\mathscr{O}_{\mathfrak{B}}(H_{\mathfrak{B}})) = 9$ we obtain $c_2 = -35\lambda + 30g - 12a + 294$ and hence by Proposition 6.9 we get

$$22\lambda - 20g - d\Delta + 12a - 176 = 0. \tag{6.20}$$

Now the conclusion follows from Lemma 6.19. □

Finally we conclude our discussion about classification with the following:

**Proposition 6.24.** *If $r = 3$, $n = 8$ and $\langle\mathfrak{B}\rangle = \mathbb{P}^8$, then one of the following cases holds:*

*(XIX)* $a = 0$, $\lambda = 12$, $g = 6$, *$\mathfrak{B}$ is a scroll $\mathbb{P}_Y(\mathscr{E})$ over a rational surface $Y$ with $K_Y^2 = 5$, $c_2(\mathscr{E}) = 8$ and $c_1^2(\mathscr{E}) = 20$;*

*(XX)* $a = 0$, $\lambda = 13$, $g = 8$, *$\mathfrak{B}$ is obtained as the blow-up of a Fano variety $X$ at a point $p \in X$, $|H_{\mathfrak{B}}| = |H_X - p|$;*

*(XXI)* $a = 1$, $\lambda = 11$, $g = 5$, *$\mathfrak{B}$ is the blow-up of $Q^3$ at 5 points $p_1, \ldots, p_5$, $|H_{\mathfrak{B}}| = |2H_{Q^3} - p_1 - \cdots - p_5|$;*

*(XXII)* $a = 1$, $\lambda = 11$, $g = 5$, *$\mathfrak{B}$ is a scroll over $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O} \oplus \mathscr{O}(-1))$;*

*(XXIII)* $a = 1$, $\lambda = 12$, $g = 7$, *$\mathfrak{B}$ is a linear section of $S^{10} \subset \mathbb{P}^{15}$;*

*(XXIV)* $a = 2$, $\lambda = 10$, $g = 4$, *$\mathfrak{B}$ is a scroll over $Q^2$;*

*(XXV)* $a = 3$, $\lambda = 9$, $g = 3$, *$\mathfrak{B}$ is a scroll over $\mathbb{P}^2$ or a quadric fibration over $\mathbb{P}^1$;*

*(XXVI)* $a = 4$, $\lambda = 8$, $g = 2$, $\mathfrak{B}$ *is a hyperplane section of* $\mathbb{P}^1 \times Q^3$*;*

*(XXVII)* $a = 6$, $\lambda = 6$, $g = 0$, $\mathfrak{B}$ *is a rational normal scroll.*

*Proof.* For $a = 6$ the statement follows from [Ion84]. The case with $a = 5$ is excluded by [Ion84] and Example 6.45. For $a = 4$ the statement follows from [Ion90]. For $a \in \{2,3\}$, by [FL94], [FL97] and [Ion86a] it follows that the abstract structure of $\mathfrak{B}$ is as asserted, or $a = 2$ and $\mathfrak{B}$ is a quadric fibration over $\mathbb{P}^1$; the last case is excluded by Lemma 6.21. For $a = 1$ the statement is just Proposition 5.25. Now we treat the cases with $a = 0$.

*Case* 6.24.1 ($a = 0, \lambda = 12$). Since $\deg(\mathfrak{B}) \leq 2\mathrm{codim}_{\mathbb{P}^8}(\mathfrak{B}) + 2$, it follows that $(\mathfrak{B}, H_{\mathfrak{B}})$ must be as in one of the cases (a),...,(h) of [Ion85, Theorem 1]. Cases (a), (d), (e), (g), (h) are of course impossible and case (c) is excluded by Lemma 6.21. If $\mathfrak{B}$ is as in case (b), by Lemma 6.23 we obtain that $\mathfrak{B}$ is a scroll over a birationally ruled surface (hence over a rational surface since $q = 0$). Now suppose that $(\mathfrak{B}, H_{\mathfrak{B}})$ is as in case (f). Thus there is a reduction $(X, H_X)$ as in one of the cases:

(f1) $X = \mathbb{P}^3$, $H_X \in |\mathscr{O}(3)|$;

(f2) $X = Q^3$, $H_X \in |\mathscr{O}(2)|$;

(f3) $X$ is a $\mathbb{P}^2$-bundle over a smooth curve such that $\mathscr{O}_X(H_X)$ induces $\mathscr{O}(2)$ on each fiber.

By definition of reduction we have $X \subset \mathbb{P}^N$, where $N = 8 + s$, $\deg(X) = \lambda + s = 12 + s$ and $s$ is the number of points blown up on $X$ to get $\mathfrak{B}$. Case (f1) and (f2) are impossible because they force $\lambda$ to be respectively 16 and 11. In case (f3), we have a projection $\beta : (X, H_X) \to (Y, H_Y)$ over a curve $Y$ such that $\beta^*(H_Y) = 2K_X + 3H_X$. Hence we get

$$K_X H_X^2 = (2K_X + 3H_X)^2 \cdot H_X/12 - K_X^2 \cdot H_X/3 - 3H_X^3/4 = -K_X^2 \cdot H_X/3 - 3H_X^3/4,$$

from which we deduce that

$$\begin{aligned} 0 &= (2K_X + 3H_X)^3 = 8K_X^3 + 36K_X^2 \cdot H_X + 54K_X \cdot H_X^2 + 27H_X^3 \\ &= 8K_X^3 + 18K_X^2 \cdot H_X - 27H_X^3/2 \\ &= 8(K_{\mathfrak{B}}^3 - 8s) + 18K_X^2 \cdot H_X - 27(\deg(\mathfrak{B}) + s)/2 \\ &= 18K_X^2 \cdot H_X - 155s/2 - 210. \end{aligned}$$

Since $s \leq 12$ (see [BB05b, Lemma 8.1]), we conclude that case (f) does not occur. Thus, $\mathfrak{B} = \mathbb{P}_Y(\mathscr{E})$ is a scroll over a surface $Y$; moreover, by Lemma 6.22 and [BS95, Theorem 11.1.2], we obtain $K_Y^2 = 5$, $c_2(\mathscr{E}) = K_Y^2 - K_S^2 = 8$ and $c_1^2(\mathscr{E}) = \lambda + c_2(\mathscr{E}) = 20$.

*Case* 6.24.2 ($a = 0, \lambda = 13$). The proof is located in [MR05, page 16], but we sketch it for the reader's convenience. By Lemma 6.19 we know that $\chi(\mathscr{O}_S) = 2$ and $K_S$ is an exceptional curve of the first kind. Thus, if we blow-down the divisor $K_S$, we obtain a $K3$-surface. By using adjunction theory (see for instance [BS95] or Ionescu's papers cited in the bibliography) and by Lemmas 6.21, 6.22 and 6.23 it follows that the adjunction map $\phi_{|K_{\mathfrak{B}}+2H_{\mathfrak{B}}|}$ is a generically finite morphism; moreover, since $(K_{\mathfrak{B}} + 2H_{\mathfrak{B}}) \cdot K_S = 0$, we see that $\phi_{|K_{\mathfrak{B}}+2H_{\mathfrak{B}}|}$ is not a finite morphism. So, we deduce that there is a $(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-1))$ inside $\mathfrak{B}$ and, after the blow-down of this divisor, we get a smooth Fano 3-fold $X \subset \mathbb{P}^9$ of sectional genus 8 and degree 14.

□

## 6.6 Examples

As in §5.3, in order to verify the calculations in the following examples, we suggest the use of [GS10] or [So11].

**Example 6.25** ($r = 1,2,3; n = 3,4,5; a = 1; d = 1$)**.** As already said in §4.1, if $Q \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ is a smooth quadric, then the linear system $|\mathscr{I}_{Q,\mathbb{P}^n}(2)|$ defines a birational transformation $\psi : \mathbb{P}^n \dashrightarrow \mathbf{S} \subset \mathbb{P}^{n+1}$ of type $(2,1)$ whose image is a smooth quadric.

**Example 6.26** ($r = 1; n = 4; a = 0; d = 3$)**.** See also [CK89]. If $X \subset \mathbb{P}^4$ is a nondegenerate curve of genus 1 and degree 5, then $X$ is the scheme-theoretic intersection of the quadrics (of rank 3) containing $X$ and $|\mathscr{I}_{X,\mathbb{P}^4}(2)|$ defines a Cremona transformation $\mathbb{P}^4 \dashrightarrow \mathbb{P}^4$ of type $(2,3)$.

**Example 6.27** ($r = 1,2,3; n = 4,5,7; a = 1,0,1; d = 2$)**.** As already said in Example 5.13, if $X \subset \mathbb{P}^n$ is either $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ or $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, then $|\mathscr{I}_{X,\mathbb{P}^n}(2)|$ defines a birational transformation $\psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of type $(2,2)$. The restriction of $\psi$ to a general hyperplane is a birational transformation $\mathbb{P}^{n-1} \dashrightarrow \mathbf{S} \subset \mathbb{P}^n$ of type $(2,2)$ and $\mathbf{S}$ is a smooth quadric.

**Example 6.28** ($r = 1; n = 4; a = 2; d = 1$ - not satisfying 6.3)**.** This essentially gives an example for case (D) of Lemma 6.12. We have a special birational transformation $\psi : \mathbb{P}^4 \dashrightarrow \mathbf{S} \subset \mathbb{P}^6$ of type $(2,1)$ with base locus $X$, image $\mathbf{S}$ and base locus of the inverse $Y$, as follows:

$$
\begin{aligned}
X &= V(x_0x_1 - x_2^2 - x_3^2, -x_0^2 - x_1^2 + x_2x_3, x_4),\\
\mathbf{S} &= V(y_2y_3 - y_4^2 - y_5^2 - y_0y_6, y_2^2 + y_3^2 - y_4y_5 + y_1y_6),\\
P_{\mathbf{S}}(t) &= (4t^4 + 24t^3 + 56t^2 + 60t + 24)/4!,\\
\operatorname{sing}(\mathbf{S}) &= V(y_6, y_5^2, y_4y_5, y_3y_5, y_2y_5, y_4^2, y_3y_4, y_2y_4, 2y_1y_4 + y_0y_5,\\
&\qquad y_0y_4 + 2y_1y_5, y_3^2, y_2y_3, y_2^2, y_1y_2 + 2y_0y_3, 2y_0y_2 + y_1y_3),\\
P_{\operatorname{sing}(\mathbf{S})}(t) &= t + 5,\\
(\operatorname{sing}(\mathbf{S}))_{\mathrm{red}} &= V(y_6, y_5, y_4, y_3, y_2),\\
Y = (Y)_{\mathrm{red}} &= (\operatorname{sing}(\mathbf{S}))_{\mathrm{red}} = V(y_6, y_5, y_4, y_3, y_2).
\end{aligned}
$$

**Example 6.29** ($r = 1,2,3; n = 4,5,6; a = 3; d = 1$)**.** See also [RS01] and [Sem31]. If $X = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5 \subset \mathbb{P}^6$, then $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a birational transformation $\psi : \mathbb{P}^6 \dashrightarrow \mathbf{S} \subset \mathbb{P}^9$ of type $(2,1)$ whose base locus is $X$ and whose image is $\mathbf{S} = \mathbb{G}(1,4)$. Restricting $\psi$ to a general $\mathbb{P}^5 \subset \mathbb{P}^6$ (resp. $\mathbb{P}^4 \subset \mathbb{P}^6$) we obtain a birational transformation $\mathbb{P}^5 \dashrightarrow \mathbf{S} \subset \mathbb{P}^8$ (resp. $\mathbb{P}^4 \dashrightarrow \mathbf{S} \subset \mathbb{P}^7$) whose image is a smooth linear section of $\mathbb{G}(1,4) \subset \mathbb{P}^9$.

**Example 6.30** ($r = 2; n = 6; a = 0; d = 4$)**.** See also [CK89] and [HKS92]. Let $Z = \{p_1, \ldots, p_8\}$ be a set of 8 points in $\mathbb{P}^2$ such that no 4 of the $p_i$ are collinear and no 7 of the $p_i$ lie on a conic and consider the blow-up $X = \mathrm{Bl}_Z(\mathbb{P}^2)$ embedded in $\mathbb{P}^6$ by $|4H_{\mathbb{P}^2} - p_1 - \cdots - p_8|$. Then the homogeneous ideal of $X$ is generated by quadrics and $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a Cremona transformation $\mathbb{P}^6 \dashrightarrow \mathbb{P}^6$ of type $(2,4)$. The same happens when $X \subset \mathbb{P}^6$ is a septic elliptic scroll with $e = -1$.

**Example 6.31** ($r = 2; n = 6; a = 1; d = 3$)**.** As already said in Examples 5.14 and 5.15, if $X \subset \mathbb{P}^6$ is a general hyperplane section of an Edge variety of dimension 3 and degree 7 in $\mathbb{P}^7$, then $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a birational transformation $\psi : \mathbb{P}^6 \dashrightarrow \mathbf{S} \subset \mathbb{P}^7$ of type $(2,3)$ whose image is a rank 6 quadric.

**Example 6.32** $(r=2; n=6; a=2; d=2)$**.** If $X\subset\mathbb{P}^6$ is the blow-up of $\mathbb{P}^2$ at 3 general points $p_1,p_2,p_3$ with $|H_X|=|3H_{\mathbb{P}^2}-p_1-p_2-p_3|$, then $\mathrm{Sec}(X)$ is a cubic hypersurface. By Fact 6.5 and 6.6 we deduce that $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a birational transformation $\psi:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^8$ and its type is $(2,2)$. The image $\mathbf{S}$ is a complete intersection of two quadrics, $\dim(\mathrm{sing}(\mathbf{S}))=1$ and the base locus of the inverse is $\mathbb{P}^2\times\mathbb{P}^2\subset\mathbb{P}^8$. Alternatively, we can obtain the transformation $\psi:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^8$ by restriction to a general $\mathbb{P}^6\subset\mathbb{P}^8$ of the special Cremona transformation $\mathbb{P}^8\dashrightarrow\mathbb{P}^8$ of type $(2,2)$.

**Example 6.33** $(r=2; n=6; a=3; d=2)$**.** See also [RS01] and [Sem31]. If $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(4))$ or $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2)\oplus\mathscr{O}(3))$, then $|\mathscr{I}_{X,\mathbb{P}^6}(2)|$ defines a birational transformations $\psi:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^9$ of type $(2,2)$ whose base locus is $X$ and whose image is $\mathbf{S}=\mathbb{G}(1,4)$.

**Example 6.34** $(r=2,3; n=6,7; a=5; d=1)$**.** See also [Zak93, III Theorem 3.8]. If $X=\mathbb{G}(1,4)\subset\mathbb{P}^9\subset\mathbb{P}^{10}$, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi:\mathbb{P}^{10}\dashrightarrow\mathbf{S}\subset\mathbb{P}^{15}$ of type $(2,1)$ whose base locus is $X$ and whose image is the spinorial variety $\mathbf{S}=S^{10}\subset\mathbb{P}^{15}$. Restricting $\psi$ to a general $\mathbb{P}^7\subset\mathbb{P}^{10}$ (resp. $\mathbb{P}^6\subset\mathbb{P}^{10}$) we obtain a special birational transformation $\mathbb{P}^7\dashrightarrow\mathbf{S}\subset\mathbb{P}^{12}$ (resp. $\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^{11}$) whose dimension of the base locus is $r=3$ (resp. $r=2$) and whose image is a linear section of $S^{10}\subset\mathbb{P}^{15}$. In the first case $\mathbf{S}=\overline{\psi(\mathbb{P}^7)}$ is smooth while in the second case the singular locus of $\mathbf{S}=\overline{\psi(\mathbb{P}^6)}$ consists of 5 lines, image of the 5 Segre 3-folds containing del Pezzo surface of degree 5 and spanned by its pencils of conics.

**Example 6.35** $(r=2,3; n=6,7; a=6; d=1)$**.** See also [RS01], [Sem31] and [Zak93, III Theorem 3.8]. We have a birational transformation $\psi:\mathbb{P}^8\dashrightarrow\mathbb{G}(1,5)\subset\mathbb{P}^{14}$ of type $(2,1)$ whose base locus is $\mathbb{P}^1\times\mathbb{P}^3\subset\mathbb{P}^7\subset\mathbb{P}^8$ and whose image is $\mathbb{G}(1,5)$. Restricting $\psi$ to a general $\mathbb{P}^7\subset\mathbb{P}^8$ we obtain a birational transformation $\mathbb{P}^7\dashrightarrow\mathbf{S}\subset\mathbb{P}^{13}$ whose base locus $X$ is a rational normal scroll and whose image $\mathbf{S}$ is a smooth linear section of $\mathbb{G}(1,5)\subset\mathbb{P}^{14}$. Restricting $\psi$ to a general $\mathbb{P}^6\subset\mathbb{P}^8$ we obtain a birational transformation $\psi=\psi|_{\mathbb{P}^6}:\mathbb{P}^6\dashrightarrow\mathbf{S}\subset\mathbb{P}^{12}$ whose base locus $X$ is a rational normal scroll (hence either $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1)\oplus\mathscr{O}(3))$ or $X=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2)\oplus\mathscr{O}(2))$) and whose image $\mathbf{S}$ is a singular linear section of $\mathbb{G}(1,5)\subset\mathbb{P}^{14}$. In this case, we denote by $Y\subset\mathbf{S}$ the base locus of the inverse of $\psi$ and by $F=(F_0,\dots,F_5):\mathbb{P}^5\dashrightarrow\mathbb{P}^5$ the restriction of $\psi$ to $\mathbb{P}^5=\mathrm{Sec}(X)$. We have

$$\begin{aligned}
Y &= \overline{\psi(\mathbb{P}^5)}=\overline{F(\mathbb{P}^5)}=\mathbb{G}(1,3)\subset\mathbb{P}^5\subset\mathbb{P}^{12},\\
J_4 &:= \left\{x=[x_0,\dots,x_5]\in\mathbb{P}^5\setminus X:\mathrm{rank}\left(\left(\partial F_i/\partial x_j(x)\right)_{i,j}\right)\le 4\right\}_{\mathrm{red}}\\
&= \left\{x=[x_0,\dots,x_5]\in\mathbb{P}^5\setminus X:\dim\left(\overline{F^{-1}\left(F(x)\right)}\right)\ge 2\right\}_{\mathrm{red}}\ \text{and}\ \dim(J_4)=3,\\
\overline{\psi(J_4)} &= (\mathrm{sing}(\mathbf{S}))_{\mathrm{red}}=\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2))\subset Y.
\end{aligned}$$

**Example 6.36.** We have a good candidate for the restriction to a general $\mathbb{P}^6\subset\mathbb{P}^8$ of a quadratic Cremona transformation of $\mathbb{P}^8$ whose base locus is a 3-fold of degree 12 and sectional genus 6. In fact, there exists a smooth nondegenerate curve $C\subset\mathbb{P}^6$ of degree 12, genus 6, and having homogeneous ideal generated by 9 quadrics. Moreover, if $\psi:\mathbb{P}^6\dashrightarrow\mathbf{Z}\subset\mathbb{P}^8$ is the rational map defined by $|\mathscr{I}_{C,\mathbb{P}^6}(2)|$, then the image $\mathbf{Z}$ has degree 14 and so $\psi$ is birational. We also have that the homogeneous ideal of $\mathbf{Z}$ is generated by quintics and sextics.

The curve $C \subset \mathbb{P}^6$ can be constructed as follows: let $Y \subset \mathbb{P}^5$ be a del Pezzo surface of degree 5. Consider the Veronese embedding $\nu_2 : \mathbb{P}^5 \to \mathbb{P}^{20}$ and let $Y' = \nu_2(Y)$. $Y'$ is a surface of degree 20, sectional genus 6 and its linear span is a $\mathbb{P}^{15} \subset \mathbb{P}^{20}$. Thus, via a sequence of 8 inner projections, we obtain a surface $S \subset \mathbb{P}^7$ of degree 12 and sectional genus 6. Now we take a smooth hyperplane section of $S$. This is what the following Macaulay2 code does.

```
i1 : ringP2=QQ[z_0..z_2];
i2 : ringP5=QQ[t_0..t_5];
i3 : g=map(ringP2,ringP5,gens image basis(3,intersect(ideal(z_0,z_1),ideal(z_1,z_2),
           ideal(z_2,z_0),ideal(z_0-z_1,z_2-z_0))));
i4 : ringP20=QQ[t_0..t_20];
i5 : v=map(ringP5,ringP20,gens (ideal vars ringP5)^2);
i6 : ringP15=ringP20/(ideal image basis(1,saturate kernel(g*v)));
i7 : p=map(ringP20,ringP15,vars ringP20);
i8 : f=g*v*p;
i9 : L=ideal image basis(1,intersect(preimage(f,ideal(z_0,z_1-z_2)),
           preimage(f,ideal(z_1,z_2-z_0)),preimage(f,ideal(z_2,z_0-z_1)),
           preimage(f,ideal(z_0,z_1+z_2)),preimage(f,ideal(z_1,z_2+z_0)),
           preimage(f,ideal(z_2,z_0+z_1)),preimage(f,ideal(z_0-z_1,z_2+z_0)),
           preimage(f,ideal(z_0+z_1,z_2-z_0))));
i10 : ringP7=QQ[x_0..x_7];
i11 : h=f*map(ringP15,ringP7,gens(L));
i12 : idealS=saturate kernel h;
i13 : H=sub(idealS, {x_7=>x_0+x_1+x_2+x_3+x_4+x_5+x_6});
i14 : ringP6=QQ[x_0..x_6];
i15 : idealC=saturate sub(H,ringP6);
i16 : C=Proj(ringP6/idealC);
i17 : dim singularLocus C
o17 = -infinity
i18 : dim C
o18 = 1
i19 : degree C
o19 = 12
i20 : genus C
o20 = 6
i21 : numgens idealC
o21 = 9
i22 : ringP8=QQ[y_0..y_8];
i23 : psi=map(ringP6,ringP8,gens idealC);
i24 : Z=Proj(coimage(psi));
i25 : dim Z
o25 = 6
i26 : degree Z
o26 = 14
i27 : DegreeOfpsi=(-5*degree(C)+2*genus(C)+62)/degree(Z)
o27 = 1
```

**Example 6.37** ($r = 3; n = 8; a = 0; d = 5$)**.** See also [HKS92]. If $\mathscr{X} \subset \mathbb{P}^9$ is a general 3-dimensional linear section of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$, $p \in \mathscr{X}$ is a general point and $X \subset \mathbb{P}^8$ is the image of $\mathscr{X}$ under the projection from $p$, then the homogeneous ideal of $X$ is generated by quadrics and $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a Cremona transformation $\mathbb{P}^8 \dashrightarrow \mathbb{P}^8$ of type $(2,5)$.

**Example 6.38** ($r = 3; n = 8; a = 1; d = 3$)**.** As already said in Example 5.17, if $X \subset \mathbb{P}^8$ is the blow-up of the smooth quadric $Q^3 \subset \mathbb{P}^4$ at 5 general points $p_1, \ldots, p_5$ with $|H_X| = |2H_{Q^3} - p_1 - \cdots - p_5|$, then $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^9$ of type $(2,3)$ whose image is a cubic hypersurface with singular locus of dimension 3.

**Example 6.39** ($r = 3; n = 8; a = 1; d = 4$ - incomplete)**.** By [AB10] (see also [BFF12]) there exists a smooth irreducible nondegenerate linearly normal 3-dimensional variety $X \subset \mathbb{P}^8$ with $h^1(X, \mathscr{O}_X) = 0$, degree $\lambda = 11$, sectional genus $g = 5$, having the structure of a scroll $\mathbb{P}_{\mathbb{F}^1}(\mathscr{E})$ with $c_1(\mathscr{E}) = 3C_0 + 5f$ and $c_2(\mathscr{E}) = 10$ and hence having degrees of the Segre classes $s_1(X) = -85$, $s_2(X) = 386$, $s_3(X) = -1330$. Now, by Fact 6.6, $X \subset \mathbb{P}^8$ is arithmetically Cohen-Macaulay and by Riemann-Roch Theorem, denoting with $C$ a general curve section of $X$, we obtain

$$h^0(\mathbb{P}^8, \mathscr{I}_X(2)) = h^0(\mathbb{P}^6, \mathscr{I}_C(2)) = h^0(\mathbb{P}^6, \mathscr{O}_{\mathbb{P}^6}(2)) - h^0(C, \mathscr{O}_C(2)) = 28 - (2\lambda + 1 - g), \quad (6.21)$$

hence $h^0(\mathbb{P}^8, \mathscr{I}_X(2)) = 10$. If the homogeneous ideal of $X$ is generated by quadratic forms[3] or at least if $X = V(H^0(\mathscr{I}_X(2)))$, the linear system $|\mathscr{I}_X(2)|$ defines a rational map $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} = \overline{\psi(\mathbb{P}^8)} \subset \mathbb{P}^9$ whose base locus is $X$ and whose image $\mathbf{S}$ is nondegenerate. Now, by (6.8) we deduce $\deg(\psi)\deg(\mathbf{S}) = 2$, from which $\deg(\psi) = 1$ and $\deg(\mathbf{S}) = 2$.

We have a good candidate for the restriction of $\psi$ to a general $\mathbb{P}^6 \subset \mathbb{P}^8$. In fact, there exists a smooth irreducible nondegenerate linearly normal curve $C \subset \mathbb{P}^6$ of degree $\lambda = 11$, genus $g = 5$, and having homogeneous ideal generated by the 10 quadrics:

$$\begin{gathered} x_5^2 - x_4x_6,\ -x_3x_6 + x_2x_6 + x_4x_5 - x_0x_1,\ x_3x_5 - x_2x_6,\ x_2x_5 - x_1x_6, \\ x_3x_4 - x_1x_6,\ x_2x_4 - x_1x_5,\ x_2x_6 - x_1x_6 + x_1x_5 - x_4^2 - x_3^2 + x_2x_3 + x_0^2, \\ x_0x_6 - x_1x_5 + x_1x_3,\ x_0x_6 - x_1x_5 + x_2^2,\ x_0x_5 - x_1x_4 + x_1x_2. \end{gathered} \quad (6.22)$$

The quadrics (6.22) give a rational map $\psi' : \mathbb{P}^6 \dashrightarrow \mathbf{S}' \subset \mathbb{P}^9$ and since $\deg(\psi')\deg(\mathbf{S}') = -5\lambda + 2g + 62 = 17$, we have that $\psi'$ is birational. Moreover, the homogeneous ideal of $\mathbf{S}'$ is generated by 6 quartics and a rank 6 quadric defined by:

$$y_3^2 - y_3y_4 + y_2y_5 + y_0y_7 - y_0y_8.$$

**Example 6.40** ($r = 3; n = 8; a = 1; d = 4$)**.** As already said in Example 5.16, if $X \subset \mathbb{P}^8$ is a general linear 3-dimensional section of the spinorial variety $S^{10} \subset \mathbb{P}^{15}$, then $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^9$ of type $(2,4)$ whose image is a smooth quadric.

**Example 6.41** ($r = 3; n = 8; a = 2; d = 3$)**.** By [FL97] (see also [BF05]) there exists a smooth irreducible nondegenerate linearly normal 3-dimensional variety $X \subset \mathbb{P}^8$ with $h^1(X, \mathscr{O}_X) = 0$, degree $\lambda = 10$, sectional genus $g = 4$, having the structure of a scroll $\mathbb{P}_{Q^2}(\mathscr{E})$ with $c_1(\mathscr{E}) = \mathscr{O}_Q(3,3)$ and $c_2(\mathscr{E}) = 8$ and hence having degrees of the Segre classes $s_1(X) = -76$, $s_2(X) = 340$, $s_3(X) = -1156$. By Fact 6.6, $X \subset \mathbb{P}^8$ is arithmetically Cohen-Macaulay and its homogeneous ideal is generated by quadratic forms. So by (6.21) we have $h^0(\mathbb{P}^8, \mathscr{I}_X(2)) = 11$ and the linear

[3] One says that a subvariety $X \subset \mathbb{P}^n$ is 2-regular (in the sense of Castelnuovo-Mumford) if $h^j(\mathbb{P}^n, \mathscr{I}_X(2-j)) = 0$ for all $j > 0$. If $X$ is 2-regular, then its homogeneous ideal is generated by forms of degrees $\leq 2$ (see [Mum66]). Now, if $X \subset \mathbb{P}^8$ is a scroll over $\mathbb{F}_1$ as above, we have $h^j(\mathbb{P}^8, \mathscr{I}_X(2-j)) = 0$ for $j > 0$ and $j \neq 4$, but unfortunately we also have $h^4(\mathbb{P}^8, \mathscr{I}_X(-2)) = h^3(X, \mathscr{O}_X(-2)) = -P_X(-2) = 5$.

system $|\mathscr{I}_X(2)|$ defines a rational map $\psi:\mathbb{P}^8 \dashrightarrow \mathbf{S}\subset\mathbb{P}^{10}$ whose base locus is $X$ and whose image $\mathbf{S}$ is nondegenerate. By (6.8) it follows that $\deg(\psi)\deg(\mathbf{S})=4$ and hence $\deg(\psi)=1$ and $\deg(\mathbf{S})=4$.

**Example 6.42** ($r=3; n=8; a=3; d=2$)**.** By [FL94] (see also [BF05]) there exists a smooth irreducible nondegenerate linearly normal 3-dimensional variety $X\subset\mathbb{P}^8$ with $h^1(X,\mathscr{O}_X)=0$, degree $\lambda=9$, sectional genus $g=3$, having the structure of a scroll $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$ with $c_1(\mathscr{E})=4$ and $c_2(\mathscr{E})=7$ and hence having degrees of the Segre classes $s_1(X)=-67$, $s_2(X)=294$, $s_3(X)=-984$. By Fact 6.6, $X\subset\mathbb{P}^8$ is arithmetically Cohen-Macaulay and its homogeneous ideal is generated by quadratic forms. So by (6.21) we have $h^0(\mathbb{P}^8,\mathscr{I}_X(2))=12$ and the linear system $|\mathscr{I}_X(2)|$ defines a rational map $\psi:\mathbb{P}^8\dashrightarrow\mathbf{S}\subset\mathbb{P}^{11}$ whose base locus is $X$ and whose image $\mathbf{S}$ is nondegenerate. By (6.8) it follows that $\deg(\psi)\deg(\mathbf{S})=8$ and in particular $\deg(\psi)\neq 0$ i.e. $\psi:\mathbb{P}^8\dashrightarrow\mathbf{S}$ is generically quasi-finite. Again by Fact 6.6 and Fact 6.5 it follows that $\psi$ is birational and hence $\deg(\mathbf{S})=8$.

Now we find the equations for such an $X$. Consider the set of 7 points in $\mathbb{P}^2$,

$$T:=\{[1,0,0],[0,1,0],[0,0,1],[1,1,0],[1,0,1],[0,1,1],[1,1,1]\}.$$

The homogeneous ideal of $T$ is generated by 3 cubics; we have $\dim|\mathscr{I}_{T,\mathbb{P}^2}(4)|=7$ and let $\nu=\nu_{|\mathscr{I}_{T,\mathbb{P}^2}(4)|}:\mathbb{P}^2\dashrightarrow\mathbb{P}^7$. The image $S=\overline{\nu(\mathbb{P}^2)}\subset\mathbb{P}^7$ is a smooth nondegenerate surface, of degree 9, sectional genus 3 and with homogeneous ideal generated by 12 quadrics. These 12 quadrics define a special quadratic birational map $\psi':\mathbb{P}^7\dashrightarrow\mathbb{P}^{11}$ whose image is a complete intersection of 4 quadrics and whose inverse can be defined by quadrics. We then consider the 8 quadrics defining $\psi'^{-1}$ and the 4 quadrics defining $\overline{\psi'(\mathbb{P}^7)}$. These 12 quadrics give a Cremona transformation of $\mathbb{P}^{11}$. Explicitly, there is a Cremona transformation $\phi:\mathbb{P}^{11}\dashrightarrow\mathbb{P}^{11}$ defined by:

$$\begin{gathered} x_6x_{10}-x_5x_{11},\ x_1x_{10}-x_4x_{10}+x_3x_{11},\ x_6x_8-x_2x_{11},\ x_5x_8-x_2x_{10},\\ x_3x_8-x_0x_{10},\ x_1x_8-x_4x_8+x_0x_{11},\ x_6x_7-x_1x_9+x_4x_9-x_4x_{11},\ x_5x_7+x_3x_9-x_4x_{10},\\ x_2x_7-x_4x_8+x_0x_9,\ x_1x_5-x_4x_5+x_3x_6,\ x_2x_3-x_0x_5,\ x_1x_2-x_2x_4+x_0x_6. \end{gathered}\tag{6.23}$$

The inverse of $\phi$ is defined by:

$$\begin{gathered} -y_5y_{10}+y_4y_{11},\ y_5y_9-y_8y_9+y_6y_{10}-y_1y_{11}+y_7y_{11},\ y_2y_{10}+y_3y_{11},\ y_4y_9-y_1y_{10},\\ -y_8y_9+y_6y_{10}+y_7y_{11},\ y_3y_9+y_0y_{10},\ y_2y_9-y_0y_{11},\ y_4y_6+y_5y_7-y_1y_8,\\ y_2y_4+y_3y_5,\ -y_3y_6+y_2y_7-y_0y_8,\ y_1y_3+y_0y_4,\ y_1y_2-y_0y_5. \end{gathered}\tag{6.24}$$

The base locus of $\phi$ (resp. $\phi^{-1}$) is a variety of dimension 6, degree 9, sectional genus 3, and the support of the singular locus is a plane $\mathbb{P}^2\subset\mathbb{P}^{11}$. In particular, by restricting the above Cremona transformation to a general $\mathbb{P}^8\subset\mathbb{P}^{11}$, we obtain an explicit example of special quadratic birational transformation $\psi:\mathbb{P}^8\dashrightarrow\mathbf{S}\subset\mathbb{P}^{11}$; that is we obtain explicit equations for a 3-fold scroll $X\subset\mathbb{P}^8$ over $\mathbb{P}^2$, of degree 9 and sectional genus 3. (For example, by restricting to the subspace $V(x_0+x_1+x_2+x_5+x_8-x_{11}, x_1+x_2+x_3+x_4+x_7-x_{10}, x_0+x_3+x_4+x_5+x_6-x_9)$, everything works fine and the image is a complete intersection with singular locus of dimension 3.)

The following Macaulay2 code computes the maps $\phi$ and $\phi^{-1}$.

```
i1 : installPackage("AdjointIdeal");
i2 : installPackage("Parametrization");
i3 : ringP2=QQ[z_0..z_2];
i4 : points=intersect(ideal(z_0,z_1),ideal(z_0,z_2),ideal(z_1,z_2),ideal(z_0,z_1-z_2),
           ideal(z_1,z_0-z_2),ideal(z_2,z_0-z_1),ideal(z_0-z_1,z_0-z_2));
i5 : parametr=gens(image(basis(4,points)));
i6 : ringP7=QQ[t_0..t_7];
i7 : idealS=saturate(kernel(map(ringP2,ringP7,parametr)));
i8 : -- You could work with the surface S,
     -- but we prefer to work with a sectional curve for efficiency reasons.
     ringP6=QQ[t_0..t_6];
i9 : idealC=sub(sub(idealS, {t_7=>0}),ringP6);
i10 : ImInv=invertBirationalMap(ideal(ringP6),gens(idealC));
i11 : ringP11=QQ[x_0..x_11];
i12 : idealX=sub(ideal(image(basis(2,saturate(ideal(ImInv#0)+ImInv#1)))),vars(ringP11));
i13 : X=Proj(ringP11/idealX);
i14 : ImInv2=invertBirationalMap(ideal(ringP11),gens(idealX));
i15 : ringP11'=QQ[y_0..y_11];
i16 : Phi=map(ringP11,ringP11',gens(idealX));
i17 : InversePhi=map(ringP11',ringP11,sub(transpose(ImInv2#0),vars(ringP11')));
```

**Example 6.43** ($r = 3; n = 8; a = 3; d = 3$)**.** By [FL94] (see also [BF05]) there exists a smooth irreducible nondegenerate linearly normal 3-dimensional variety $X \subset \mathbb{P}^8$ with $h^1(X, \mathscr{O}_X) = 0$, degree $\lambda = 9$, sectional genus $g = 3$, having the structure of a quadric fibration over $\mathbb{P}^1$ and hence having degrees of the Segre classes $s_1(X) = -67$, $s_2(X) = 295$, $s_3(X) = -997$. By Fact 6.6, $X \subset \mathbb{P}^8$ is arithmetically Cohen-Macaulay and its homogeneous ideal is generated by quadratic forms. So by (6.21) we have $h^0(\mathbb{P}^8, \mathscr{I}_X(2)) = 12$ and the linear system $|\mathscr{I}_X(2)|$ defines a rational map $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{11}$ whose base locus is $X$ and whose image $\mathbf{S}$ is nondegenerate. By (6.8) it follows that $\deg(\psi)\deg(\mathbf{S}) = 5$ and hence $\deg(\psi) = 1$ and $\deg(\mathbf{S}) = 5$.

Now we find the equations for such an $X$. Consider the rational normal scroll $S(1,1,1,2) = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(1) \oplus \mathscr{O}(1) \oplus \mathscr{O}(2)) \subset \mathbb{P}^8$, which is defined by the $2 \times 2$ minors of the matrix:

$$\begin{pmatrix} x_0 & x_2 & x_4 & x_6 & x_7 \\ x_1 & x_3 & x_5 & x_7 & x_8 \end{pmatrix}$$

Intersect $S(1,1,1,2)$ with the quadric $Q \subset \mathbb{P}^8$ defined by:

$$x_3^2 + x_3x_4 + x_0x_5 + x_1x_5 + x_2x_5 + x_3x_5 + x_1x_6 + x_1x_7 + x_6x_7 + x_7^2 + x_1x_8 + x_7x_8.$$

We have $S(1,1,1,2) \cap Q = X \cup P$, where $P$ is the linear variety $V(x_8, x_7, x_5, x_3, x_1)$, while $X \subset \mathbb{P}^8$ is a nondegenerate smooth variety of degree 9, sectional genus 3 and having homogeneous ideal generated by 12 quadrics. These 12 quadrics give the birational map $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{11}$ defined by:

$$\begin{gathered} x_7^2 - x_6x_8,\ x_5x_7 - x_4x_8,\ x_3x_7 - x_2x_8,\ x_1x_7 - x_0x_8, \\ x_5x_6 - x_4x_7,\ x_3x_6 - x_2x_7,\ x_1x_6 - x_0x_7,\ x_3x_4 - x_2x_5,\ x_1x_4 - x_0x_5, \\ x_3^2 + x_0x_5 + x_1x_5 + 2x_2x_5 + x_3x_5 + x_0x_7 + x_6x_7 + x_0x_8 + x_1x_8 + x_6x_8 + x_7x_8, \\ x_2x_3 + x_0x_4 + 2x_2x_4 + x_0x_5 + x_2x_5 + x_0x_6 + x_6^2 + x_0x_7 + x_6x_7 + x_0x_8 + x_6x_8, \\ x_1x_2 - x_0x_3. \end{gathered} \tag{6.25}$$

The image $\mathbf{S}$ is defined by the quadrics:

$$\begin{gathered} y_6y_7-y_5y_8+y_4y_{11},\ y_3y_7-y_2y_8+y_1y_{11},\ y_3y_5-y_2y_6+y_0y_{11},\\ y_3y_4-y_1y_6+y_0y_8,\ y_2y_4-y_1y_5+y_0y_7. \end{gathered} \tag{6.26}$$

It coincides with the cone over $\mathbb{G}(1,4)\subset V(y_9,y_{10})\simeq\mathbb{P}^9\subset\mathbb{P}^{11}$.

**Example 6.44** ($r=3;n=8;a=4;d=2$)**.** Consider the composition

$$f:\mathbb{P}^1\times\mathbb{P}^3\longrightarrow\mathbb{P}^1\times Q^3\subset\mathbb{P}^1\times\mathbb{P}^4\longrightarrow\mathbb{P}^9,$$

where the first map is the identity of $\mathbb{P}^1$ multiplied by $[z_0,z_1,z_2,z_3]\mapsto[z_0^2,z_0z_1,z_0z_2,z_0z_3,z_1^2+z_2^2+z_3^2]$, and the last map is $([t_0,t_1],[y_0,\ldots,y_4])\mapsto[t_0y_0,\ldots,t_0y_4,t_1y_0,\ldots,t_1y_4]=[x_0,\ldots,x_9]$. In the equations defining $\overline{f(\mathbb{P}^1\times\mathbb{P}^3)}\subset\mathbb{P}^9$, by replacing $x_9$ with $x_0$, we obtain the quadrics:

$$\begin{gathered} -x_0x_3+x_4x_8,\ -x_0x_2+x_4x_7,\ x_3x_7-x_2x_8,\ -x_0x_5+x_6^2+x_7^2+x_8^2,\ -x_0x_1+x_4x_6,\\ x_3x_6-x_1x_8,\ x_2x_6-x_1x_7,\ -x_0^2+x_1x_6+x_2x_7+x_3x_8,\ -x_0^2+x_4x_5,\ x_3x_5-x_0x_8,\\ x_2x_5-x_0x_7,\ x_1x_5-x_0x_6,\ x_1^2+x_2^2+x_3^2-x_0x_4. \end{gathered} \tag{6.27}$$

Denoting with $I$ the ideal generated by the quadrics (6.27) and $X=V(I)$, we have that $I$ is saturated and $X$ is smooth. The linear system $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi:\mathbb{P}^8\dashrightarrow\mathbf{S}\subset\mathbb{P}^{12}$ whose base locus is $X$ and whose image is the variety $\mathbf{S}$ with homogeneous ideal generated by:

$$\begin{gathered} y_6y_9-y_5y_{10}+y_2y_{11},\ y_6y_8-y_4y_{10}+y_1y_{11},\ y_5y_8-y_4y_9+y_0y_{11},\ y_2y_8-y_1y_9+y_0y_{10},\\ y_2y_4-y_1y_5+y_0y_6,\ y_2^2+y_5^2+y_6^2+y_7^2-y_7y_8+y_0y_9+y_1y_{10}+y_4y_{11}-y_3y_{12}. \end{gathered} \tag{6.28}$$

We have $\deg(\mathbf{S})=10$ and $\dim(\operatorname{sing}(\mathbf{S}))=3$. The inverse of $\psi:\mathbb{P}^8\dashrightarrow\mathbf{S}$ is defined by:

$$\begin{gathered} -y_7y_8+y_0y_9+y_1y_{10}+y_4y_{11},\ y_0y_5+y_1y_6-y_4y_7-y_{11}y_{12},\ y_0y_2-y_4y_6-y_1y_7-y_{10}y_{12},\\ -y_1y_2-y_4y_5-y_0y_7-y_9y_{12},\ -y_0^2-y_1^2-y_4^2-y_8y_{12},\ -y_3y_8-y_9^2-y_{10}^2-y_{11}^2,\\ -y_3y_4-y_5y_9-y_6y_{10}-y_7y_{11},\ -y_1y_3-y_2y_9-y_7y_{10}+y_6y_{11},\ -y_0y_3-y_7y_9+y_2y_{10}+y_5y_{11}. \end{gathered} \tag{6.29}$$

Note that $\mathbf{S}\subset\mathbb{P}^{12}$ is the intersection of a quadric hypersurface in $\mathbb{P}^{12}$ with the cone over $\mathbb{G}(1,4)\subset\mathbb{P}^9\subset\mathbb{P}^{12}$.

**Example 6.45** ($r=3;n=8;a=5$ - with non liftable inverse)**.** If $X\subset\mathbb{P}^8$ is the blow-up of $\mathbb{P}^3$ at a point $p$ with $|H_X|=|2H_{\mathbb{P}^3}-p|$, then (modulo a change of coordinates) the homogeneous ideal of $X$ is generated by the quadrics:

$$\begin{gathered} x_6x_7-x_5x_8,\ x_3x_7-x_2x_8,\ x_5x_6-x_4x_8,\ x_2x_6-x_1x_8,\ x_5^2-x_4x_7,\ x_3x_5-x_1x_8,\ x_2x_5-x_1x_7,\\ x_3x_4-x_1x_6,\ x_2x_4-x_1x_5,\ x_2x_3-x_0x_8,\ x_1x_3-x_0x_6,\ x_2^2-x_0x_7,\ x_1x_2-x_0x_5,\ x_1^2-x_0x_4. \end{gathered} \tag{6.30}$$

The linear system $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi:\mathbb{P}^8\dashrightarrow\mathbb{P}^{13}$ whose base locus is $X$ and whose image is the variety $\mathbf{S}$ with homogeneous ideal generated by:

$$\begin{gathered} y_8y_{10}-y_7y_{12}-y_3y_{13}+y_5y_{13},\ y_8y_9+y_6y_{10}-y_7y_{11}-y_3y_{12}+y_1y_{13},\ y_6y_9-y_5y_{11}+y_1y_{12},\\ y_6y_7-y_5y_8-y_4y_{10}+y_2y_{12}-y_0y_{13},\ y_3y_6-y_5y_6+y_1y_8+y_4y_9-y_2y_{11}+y_0y_{12},\\ y_3y_4-y_2y_6+y_0y_8,\ y_3^2y_5-y_3y_5^2+y_1y_3y_7-y_2y_3y_9+y_2y_5y_9-y_0y_7y_9-y_1y_2y_{10}+y_0y_5y_{10}. \end{gathered} \tag{6.31}$$

We have $\deg(\mathbf{S}) = 19$, $\dim(\mathrm{sing}(\mathbf{S})) = 4$ and the degrees of Segre classes of $X$ are: $s_1 = -49$, $s_2 = 201$, $s_3 = -627$. So, by (6.9), we deduce that the inverse of $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S}$ is not liftable; however, a representative of the equivalence class of $\psi^{-1}$ is defined by:

$$\begin{gathered} y_{12}^2 - y_{11}y_{13},\ y_8y_{12} - y_6y_{13},\ y_8y_{11} - y_6y_{12},\ -y_6y_{10} + y_7y_{11} + y_3y_{12} - y_5y_{12},\ y_8^2 - y_4y_{13}, \\ y_6y_8 - y_4y_{12},\ y_3y_8 - y_2y_{12} + y_0y_{13},\ y_6^2 - y_4y_{11},\ y_5y_6 - y_1y_8 - y_4y_9. \end{gathered} \tag{6.32}$$

We also point out that $\mathrm{Sec}(X)$ has dimension 6 and degree 6 (against Proposition 6.4).

**Example 6.46** ($r = 3; n = 8; a = 6; d = 2$)**.** See also [RS01] and [Sem31]. If $X = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(1) \oplus \mathscr{O}(4))$ or $X = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(2) \oplus \mathscr{O}(3))$ or $X = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2) \oplus \mathscr{O}(2) \oplus \mathscr{O}(2))$, then the linear system $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{14}$ of type $(2,2)$ whose base locus is $X$ and whose image is $\mathbf{S} = \mathbb{G}(1,5)$.

**Example 6.47** ($r = 3; n = 8; a = 7; d = 1$)**.** See also [CMR04, Example 2.7] and [Ion90]. Let $Z = \{p_1, \dots, p_8\} \subset \mathbb{P}^2$ be such that no 4 of the $p_i$ are collinear and no 7 of the $p_i$ lie on a conic and consider the scroll $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E}) \subset \mathbb{P}^7$ associated to the very ample vector bundle $\mathscr{E}$ of rank 2, given as an extension by the following exact sequence $0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{E} \to \mathscr{I}_{Z,\mathbb{P}^2}(4) \to 0$. The homogeneous ideal of $X \subset \mathbb{P}^7$ is generated by 7 quadrics and so the linear system $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{15}$ of type $(2,1)$. Since we have $c_1(X) = 12$, $c_2(X) = 15$, $c_3(X) = 6$, we deduce $s_1(\mathscr{N}_{X,\mathbb{P}^8}) = -60$, $s_2(\mathscr{N}_{X,\mathbb{P}^8}) = 267$, $s_3(\mathscr{N}_{X,\mathbb{P}^8}) = -909$, and hence $\deg(\mathbf{S}) = 29$, by (6.8). The base locus of the inverse of $\psi$ is $\psi(\mathbb{P}^7) \simeq \mathbb{P}^6 \subset \mathbf{S} \subset \mathbb{P}^{15}$. We also observe that the restriction of $\psi|_{\mathbb{P}^7} : \mathbb{P}^7 \dashrightarrow \mathbb{P}^6$ to a general hyperplane $H \simeq \mathbb{P}^6 \subset \mathbb{P}^7$ gives rise to a transformation as in Example 6.30.

**Example 6.48** ($r = 3; n = 8; a = 8,9; d = 1$)**.** If $X \subset \mathbb{P}^7 \subset \mathbb{P}^8$ is a 3-dimensional Edge variety of degree 7 (resp. degree 6), then $|\mathscr{I}_{X,\mathbb{P}^8}(2)|$ defines a birational transformation $\mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{16}$ (resp. $\mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{17}$) of type $(2,1)$ whose base locus is $X$ and whose degree of the image is $\deg(\mathbf{S}) = 33$ (resp. $\deg(\mathbf{S}) = 38$). For memory overflow problems, we were not able to calculate the scheme $\mathrm{sing}(\mathbf{S})$; however, it is easy to obtain that $1 \leq \dim(\mathrm{sing}(\mathbf{S})) < \dim(Y) = 6$ and $\dim(\mathrm{sing}(Y)) = 1$, where $Y$ denotes the base locus of the inverse.

**Example 6.49** ($r = 3; n = 8; a = 10; d = 1$)**.** See also [RS01], [Sem31] and [Zak93, III Theorem 3.8]. We have a birational transformation $\mathbb{P}^{10} \dashrightarrow \mathbb{G}(1,6) \subset \mathbb{P}^{20}$ of type $(2,1)$ whose base locus is $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9 \subset \mathbb{P}^{10}$ and whose image is $\mathbb{G}(1,6)$. Restricting it to a general $\mathbb{P}^8 \subset \mathbb{P}^{10}$ we obtain a birational transformation $\psi : \mathbb{P}^8 \dashrightarrow \mathbf{S} \subset \mathbb{P}^{18}$ whose base locus $X$ is a rational normal scroll (hence either $X = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(1) \oplus \mathscr{O}(3))$ or $X = \mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(2) \oplus \mathscr{O}(2)))$ and whose image $\mathbf{S}$ is a linear section of $\mathbb{G}(1,6) \subset \mathbb{P}^{20}$. We denote by $Y \subset \mathbf{S}$ the base locus of the inverse of $\psi$ and by $F = (F_0, \dots, F_9) : \mathbb{P}^7 \dashrightarrow \mathbb{P}^9$ the restriction of $\psi$ to $\mathbb{P}^7 = \mathrm{Sec}(X)$. We have

$$\begin{aligned} Y &= \overline{\psi(\mathbb{P}^7)} = \overline{F(\mathbb{P}^7)} = \mathbb{G}(1,4) \subset \mathbb{P}^9 \subset \mathbb{P}^{18}, \\ J_6 &:= \left\{ x = [x_0, \dots, x_7] \in \mathbb{P}^7 \setminus X : \mathrm{rank}\left( (\partial F_i / \partial x_j(x))_{i,j} \right) \leq 6 \right\}_{\mathrm{red}} \\ &= \left\{ x = [x_0, \dots, x_7] \in \mathbb{P}^7 \setminus X : \dim\left( \overline{F^{-1}(F(x))} \right) \geq 2 \right\}_{\mathrm{red}} \text{ and } \dim(J_6) = 5, \\ \overline{\psi(J_6)} &= (\mathrm{sing}(\mathbf{S}))_{\mathrm{red}} \subset Y \text{ and } \dim\left( \overline{\psi(J_6)} \right) = 3. \end{aligned}$$

## 6.7 Summary results

**Theorem 6.50.** *Table 6.1 classifies all special quadratic transformations $\varphi$ as in §6.1 and with $r \leq 3$.*

As a consequence, we generalize Corollary 5.28.

**Corollary 6.51.** *Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{S} \subseteq \mathbb{P}^{n+a}$ be as in §6.1. If $\varphi$ is of type $(2,3)$ and $\mathbf{S}$ has coindex $c = 2$, then $n = 8$, $r = 3$ and one of the following cases holds:*

- $\Delta = 3$, $a = 1$, $\lambda = 11$, $g = 5$, $\mathfrak{B}$ *is the blow-up of* $Q^3$ *at* 5 *points;*
- $\Delta = 4$, $a = 2$, $\lambda = 10$, $g = 4$, $\mathfrak{B}$ *is a scroll over* $Q^2$*;*
- $\Delta = 5$, $a = 3$, $\lambda = 9$, $g = 3$, $\mathfrak{B}$ *is a quadric fibration over* $\mathbb{P}^1$.

*Proof.* We have that $\mathfrak{B} \subset \mathbb{P}^n$ is a *QEL*-variety of type $\delta = (r-d-c+2)/d = (r-3)/3$ and $n = ((2d-1)r+3d+c-2)/d = (5r+9)/3$. From Divisibility Theorem (Theorem 2.14), we deduce $(r,n,\delta) \in \{(3,8,0),(6,13,1),(9,18,2)\}$ and from the classification of *CC*-manifolds (Theorem 2.19), we obtain $(r,n,\delta) = (3,8,0)$. Now we apply the results in §6.5. □

In the same fashion, one can prove the following:

**Proposition 6.52.** *Let $\varphi$ be as in §6.1 and of type $(2,1)$. If $c = 2$, then $r \geq 1$ and $\mathfrak{B}$ is $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ or one of its linear sections. If $c = 3$, then $r \geq 2$ and $\mathfrak{B}$ is either $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ or $\mathbb{G}(1,4) \subset \mathbb{P}^9$ or one of their linear sections. If $c = 4$, then $r \geq 3$ and $\mathfrak{B}$ is either an OADP 3-fold in $\mathbb{P}^7$ or $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$ or one of its hyperplane sections.*

*Remark* 6.53. Imitating the proof of Proposition 5.30 (resp. Proposition 5.31), one can also compute all possible Hilbert polynomials and Hilbert schemes of lines through a general point of the base locus of a special quadratic birational transformation of type $(2,2)$ into a complete intersection of two quadrics (resp. three quadrics).

In Table 6.1 we use the following shortcuts:

$\exists^*$ flags cases for which is known a transformation $\varphi$ with base locus $\mathfrak{B}$ as required, but we do not know if the image $\mathbf{S}$ satisfies all the assumptions in §6.1;

$\exists^{**}$ flags cases for which is known that there is a smooth irreducible variety $X \subset \mathbb{P}^n$ such that, if $X = V(H^0(\mathscr{I}_X(2)))$, then the linear system $|\mathscr{I}_X(2)|$ defines a birational transformation $\varphi : \mathbb{P}^n \dashrightarrow \mathbf{S} = \overline{\varphi(\mathbb{P}^n)} \subset \mathbb{P}^{n+a}$ as stated;

? flags cases for which we do not know if there exists at least an abstract variety $\mathfrak{B}$ having the structure and the invariants required;

$\exists$ flags cases for which everything works fine.

| $r$ | $n$ | $a$ | $\lambda$ | $g$ | Abstract structure of $\mathfrak{B}$ | $d$ | $\Delta$ | $c$ | Existence |
|---|---|---|---|---|---|---|---|---|---|
| | 3 | 1 | 2 | 0 | $v_2(\mathbb{P}^1) \subset \mathbb{P}^2$ | 1 | 2 | 1 | $\exists$ Ex. 6.25 |
| 1 | 4 | 0 | 5 | 1 | Elliptic curve | 3 | 1 | 0 | $\exists$ Ex. 6.26 |
| | 4 | 1 | 4 | 0 | $v_4(\mathbb{P}^1) \subset \mathbb{P}^4$ | 2 | 2 | 1 | $\exists$ Ex. 6.27 |
| | 4 | 3 | 3 | 0 | $v_3(\mathbb{P}^1) \subset \mathbb{P}^3$ | 1 | 5 | 2 | $\exists$ Ex. 6.29 |
| | 4 | 1 | 2 | 0 | $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ | 1 | 2 | 1 | $\exists$ Ex. 6.25 |
| | 5 | 0 | 4 | 0 | $v_2(\mathbb{P}^2) \subset \mathbb{P}^5$ | 2 | 1 | 0 | $\exists$ Ex. 6.27 |
| | 5 | 3 | 3 | 0 | Hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ | 1 | 5 | 2 | $\exists$ Ex. 6.29 |
| | 6 | 0 | 7 | 1 | Elliptic scroll $\mathbb{P}_C(\mathscr{E})$ with $e(\mathscr{E}) = -1$ | 4 | 1 | 0 | $\exists$ Ex. 6.30 |
| | 6 | 0 | 8 | 3 | Blow-up of $\mathbb{P}^2$ at 8 points $p_1, \dots, p_8$, $\lvert H_{\mathfrak{B}} \rvert = \lvert 4H_{\mathbb{P}^2} - p_1 - \cdots - p_8 \rvert$ | 4 | 1 | 0 | $\exists$ Ex. 6.30 |
| 2 | 6 | 1 | 7 | 2 | Blow-up of $\mathbb{P}^2$ at 6 points $p_0, \dots, p_5$, $\lvert H_{\mathfrak{B}} \rvert = \lvert 4H_{\mathbb{P}^2} - 2p_0 - p_1 - \cdots - p_5 \rvert$ | 3 | 2 | 1 | $\exists$ Ex. 6.31 |
| | 6 | 2 | 6 | 1 | Blow-up of $\mathbb{P}^2$ at 3 points $p_1, p_2, p_3$, $\lvert H_{\mathfrak{B}} \rvert = \lvert 3H_{\mathbb{P}^2} - p_1 - p_2 - p_3 \rvert$ | 2 | 4 | 2 | $\exists$ Ex. 6.32 |
| | 6 | 3 | 5 | 0 | $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(4))$ or $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2) \oplus \mathscr{O}(3))$ | 2 | 5 | 2 | $\exists$ Ex. 6.33 |
| | 6 | 5 | 5 | 1 | Blow-up of $\mathbb{P}^2$ at 4 points $p_1 \dots, p_4$, $\lvert H_{\mathfrak{B}} \rvert = \lvert 3H_{\mathbb{P}^2} - p_1 - \cdots - p_4 \rvert$ | 1 | 12 | 3 | $\exists$ Ex. 6.34 |
| | 6 | 6 | 4 | 0 | $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(3))$ or $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(2) \oplus \mathscr{O}(2))$ | 1 | 14 | 3 | $\exists$ Ex. 6.35 |
| | 5 | 1 | 2 | 0 | $Q^3 \subset \mathbb{P}^4$ | 1 | 2 | 1 | $\exists$ Ex. 6.25 |
| | 6 | 3 | 3 | 0 | $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ | 1 | 5 | 2 | $\exists$ Ex. 6.29 |
| | 7 | 1 | 6 | 1 | Hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ | 2 | 2 | 1 | $\exists$ Ex. 6.27 |
| | 7 | 5 | 5 | 1 | Linear section of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ | 1 | 12 | 3 | $\exists$ Ex. 6.34 |
| | 7 | 6 | 4 | 0 | $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O}(1) \oplus \mathscr{O}(1) \oplus \mathscr{O}(2))$ | 1 | 14 | 3 | $\exists$ Ex. 6.35 |
| | 8 | 0 | 12 | 6 | Scroll $\mathbb{P}_Y(\mathscr{E})$, $Y$ rational surface, $K_Y^2 = 5$, $c_2(\mathscr{E}) = 8$, $c_1^2(\mathscr{E}) = 20$ | 5 | 1 | 0 | ? Ex. 6.36 |
| | 8 | 0 | 13 | 8 | Variety obtained as the projection of a Fano variety $X$ from a point $p \in X$ | 5 | 1 | 0 | $\exists$ Ex. 6.37 |
| | 8 | 1 | 11 | 5 | Blow-up of $Q^3$ at 5 points $p_1, \dots, p_5$, $\lvert H_{\mathfrak{B}} \rvert = \lvert 2H_{Q^3} - p_1 - \cdots - p_5 \rvert$ | 3 | 3 | 2 | $\exists$ Ex. 6.38 |
| 3 | 8 | 1 | 11 | 5 | Scroll over $\mathbb{P}_{\mathbb{P}^1}(\mathscr{O} \oplus \mathscr{O}(-1))$ | 4 | 2 | 1 | $\exists^{**}$ Ex. 6.39 |
| | 8 | 1 | 12 | 7 | Linear section of $S^{10} \subset \mathbb{P}^{15}$ | 4 | 2 | 1 | $\exists$ Ex. 6.40 |
| | 8 | 2 | 10 | 4 | Scroll over $Q^2$ | 3 | 4 | 2 | $\exists^*$ Ex. 6.41 |
| | 8 | 3 | 9 | 3 | Scroll over $\mathbb{P}^2$ | 2 | 8 | 3 | $\exists$ Ex. 6.42 |
| | 8 | 3 | 9 | 3 | Quadric fibration over $\mathbb{P}^1$ | 3 | 5 | 2 | $\exists^*$ Ex. 6.43 |
| | 8 | 4 | 8 | 2 | Hyperplane section of $\mathbb{P}^1 \times Q^3$ | 2 | 10 | 3 | $\exists^*$ Ex. 6.44 |
| | 8 | 6 | 6 | 0 | Rational normal scroll | 2 | 14 | 3 | $\exists$ Ex. 6.46 |
| | 8 | 7 | 8 | 3 | $\mathbb{P}_{\mathbb{P}^2}(\mathscr{E})$, where $0 \to \mathscr{O}_{\mathbb{P}^2} \to$ $\to \mathscr{E} \to \mathscr{I}_{\{p_1,\dots,p_8\},\mathbb{P}^2}(4) \to 0$ | 1 | 29 | 4 | $\exists^*$ Ex. 6.47 |
| | 8 | 8 | 7 | 2 | Edge variety | 1 | 33 | 4 | $\exists^*$ Ex. 6.48 |
| | 8 | 9 | 6 | 1 | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ | 1 | 38 | 4 | $\exists^*$ Ex. 6.48 |
| | 8 | 10 | 5 | 0 | Rational normal scroll | 1 | 42 | 4 | $\exists$ Ex. 6.49 |

Table 6.1: All transformations $\varphi$ as in §6.1 and with $r \leq 3$

# Appendix B

# Towards the study of special quadratic birational transformations whose base locus has dimension four

In this appendix we shall keep the notation of Chap. 6. We treat the case in which the dimension of the base locus is $r = 4$, although, when $\delta = 0$, we are well away from having an exhaustive classification.

## B.1 Easy cases

Similarly to Propositions 6.14 and 6.18 we deduce Proposition B.1 from the theory of *LQEL*-varieties; specifically, from Proposition 2.16, Theorems 2.18 and 2.19.

**Proposition B.1.** *Let Assumptions 6.1, 6.2 and 6.3 be valid. If $r = 4$, then either $n = 10$, $d \geq 2$, $\langle\mathfrak{B}\rangle = \mathbb{P}^{10}$, or one of the following cases holds:*

*(I)* $n = 6$, $d = 1$, $\delta = 4$, $\mathfrak{B} = Q^4 \subset \mathbb{P}^5$ *is a quadric;*

*(II)* $n = 8$, $d = 1$, $\delta = 2$, $\mathfrak{B} \subset \mathbb{P}^7$ *is either* $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ *or a linear section of* $\mathbb{G}(1,4) \subset \mathbb{P}^9$*;*

*(III)* $n = 8$, $d = 2$, $\delta = 2$, $\mathfrak{B}$ *is* $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$*;*

*(IV)* $n = 9$, $d = 1$, $\delta = 1$, $\mathfrak{B}$ *is a hyperplane section of* $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$*;*

*(V)* $n = 10$, $d = 1$, $\delta = 0$, $\mathfrak{B} \subset \mathbb{P}^9$ *is an OADP-variety.*

*Remark* B.2. Note that in Proposition B.1, all cases with $\delta > 0$ really occur (see §6.6); when $\delta = 0$, an example is obtained by taking a general 4-dimensional linear section of $\mathbb{P}^1 \times \mathbb{P}^5 \subset \mathbb{P}^{11} \subset \mathbb{P}^{12}$. Unfortunately, the classification of *OADP* 4-folds in $\mathbb{P}^{10}$ is not known; so we cannot be more precise.

## B.2 Hard cases

In this section we keep Assumption 6.1, but more generally assume that the image $\mathbf{S}$ is nondegenerate, normal and linearly normal (not necessarily factorial); we do not require Assumptions 6.2 and 6.3. As noted earlier, we have $P_{\mathfrak{B}}(1) = 11$ and $P_{\mathfrak{B}}(2) = 55 - a$ and hence

$$P_{\mathfrak{B}}(t) = \lambda \binom{t}{4} + (3\lambda + 1 - g) \binom{t}{3} + (\chi(\mathscr{O}_{\mathfrak{B}}) - a + 33) \binom{t}{2} + (11 - \chi(\mathscr{O}_{\mathfrak{B}}))t + \chi(\mathscr{O}_{\mathfrak{B}}).$$

**Proposition B.3.** *If $r = 4$, $n = 10$ and $\langle \mathfrak{B} \rangle = \mathbb{P}^{10}$, then one of the following cases holds:*

*(VI)* $a = 10$, $\lambda = 7$, $g = 0$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$, $\mathfrak{B}$ *is a rational normal scroll;*

*(VII)* $a = 7$, $\lambda = 10$, $g = 3$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$, $\mathfrak{B}$ *is either*

- *a hyperplane section of* $\mathbb{P}^1 \times Q^4 \subset \mathbb{P}^{11}$ *or*
- $\mathbb{P}(\mathscr{T}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1)) \subset \mathbb{P}^{10}$*;*

*(VIII)* $a = 6$, $\lambda = 11$, $g = 4$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$, $\mathfrak{B}$ *is a quadric fibration over* $\mathbb{P}^1$*;*

*(IX)* $a = 5$, $\lambda = 12$, $g = 5$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$, $\mathfrak{B}$ *is one of the following:*

- $\mathbb{P}^4$ *blown up at* $4$ *points* $p_1 \ldots, p_4$ *embedded by* $|2H_{\mathbb{P}^4} - p_1 - \cdots - p_4|$,
- *a scroll over a ruled surface,*
- *a quadric fibration over* $\mathbb{P}^1$*;*

*(X)* $a = 4$, $\lambda = 14$, $g = 8$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$, $\mathfrak{B}$ *is either*

- *a linear section of* $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$ *or*
- *the product of* $\mathbb{P}^1$ *with a Fano variety of even index;*

*(XI)* $a = 4$, $\lambda = 13$, $g = 6$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$, $\mathfrak{B}$ *is either*

- *a scroll over a birationally ruled surface or*
- *a quadric fibration over* $\mathbb{P}^1$*;*

*(XII)* $a = 3$, $14 \leq \lambda \leq 16$, $g \leq 11$, $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 18)/3$*;*

*(XIII)* $a = 2$, $15 \leq \lambda \leq 18$, $g \leq 14$, $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 19)/3$*;*

*(XIV)* $a = 1$, $15 \leq \lambda \leq 20$, $g \leq 17$, $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 20)/3$*;*

*(XV)* $a = 0$, $15 \leq \lambda$.

*Proof.* Denote by $\Lambda \subsetneq C \subsetneq S \subsetneq X \subsetneq \mathfrak{B}$ a sequence of general linear sections of $\mathfrak{B}$ and put $h_{\Lambda}(2) := h^0(\mathbb{P}^6, \mathscr{O}(2)) - h^0(\mathbb{P}^6, \mathscr{I}_{\Lambda}(2))$. Since $C$ is a nondegenerate curve in $\mathbb{P}^7$, we have $\lambda \geq 7$. By Castelnuovo's argument (Lemma 3.2), it follows that

$$7 \leq \min\{\lambda, 13\} \leq h_{\Lambda}(2) \leq 28 - h^0(\mathbb{P}^{10}, \mathscr{I}_{\mathfrak{B}}(2)) = 17 - a \tag{B.1}$$

and in particular we have $a \leq 10$. Moreover

- if $\lambda \geq 13$, then $h_\Lambda(2) \geq 13$ and $a \leq 4$, by (B.1);
- if $\lambda \geq 15$, then $h_\Lambda(2) \geq 14$ and $a \leq 3$, by Proposition 3.3;
- if $\lambda \geq 17$, then $h_\Lambda(2) \geq 15$ and $a \leq 2$, by Theorem 3.9;
- if $\lambda \geq 19$, then $h_\Lambda(2) \geq 16$ and $a \leq 1$, by Theorem 3.11;
- if $\lambda \geq 21$, then $h_\Lambda(2) \geq 17$ and $a = 0$, by Theorem 3.10(b).

According to the above statements, we consider the refinement $\theta = \theta(\lambda)$ of Castelnuovo's bound $\rho = \rho(\lambda)$, contained in Theorem 3.5. So, we have

$$K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^3 = 2g - 2 - 3\lambda \leq 2\theta(\lambda) - 2 - 3\lambda \leq 2\rho(\lambda) - 2 - 3\lambda. \tag{B.2}$$

Now, if $t \geq 1$, by Kodaira Vanishing Theorem and Serre Duality, it follows that $P_{\mathfrak{B}}(-t) = h^4(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}(-t)) = h^0(\mathfrak{B}, K_{\mathfrak{B}} + tH_{\mathfrak{B}})$; hence, if $P_{\mathfrak{B}}(-t) \neq 0$, then $K_{\mathfrak{B}} + tH_{\mathfrak{B}}$ is an effective divisor and we have either $K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^3 > -tH_{\mathfrak{B}}^4 = -t\lambda$ or $K_{\mathfrak{B}} \sim -tH_{\mathfrak{B}}$. Thus, by (B.2) and straightforward calculation, we deduce (see Figure B.1):

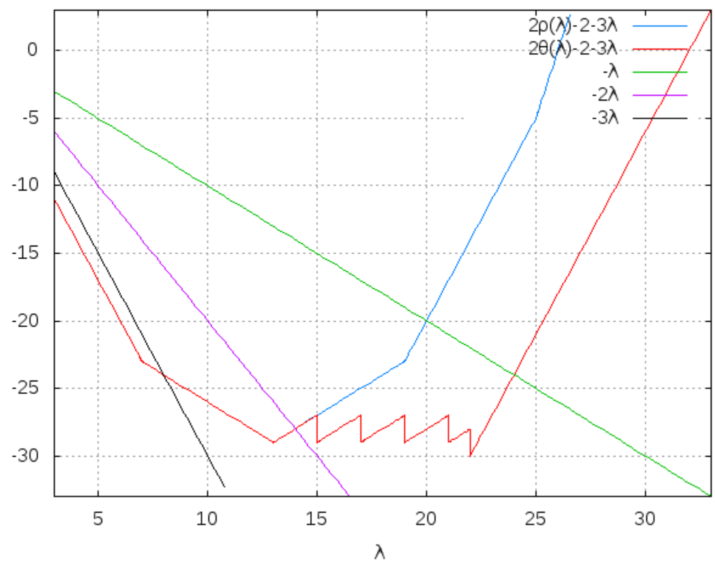


Figure B.1: Upper bounds of $K_{\mathfrak{B}} \cdot H_{\mathfrak{B}}^3$

(B.3.a) if $\lambda \leq 8$, then either $P_{\mathfrak{B}}(-3) = P_{\mathfrak{B}}(-2) = P_{\mathfrak{B}}(-1) = 0$ or $\lambda = 8$ and $K_{\mathfrak{B}} \sim -3H_{\mathfrak{B}}$;

(B.3.b) if $\lambda \leq 14$, then either $P_{\mathfrak{B}}(-2) = P_{\mathfrak{B}}(-1) = 0$ or $\lambda = 14$ and $K_{\mathfrak{B}} \sim -2H_{\mathfrak{B}}$;

(B.3.c) if $\lambda \leq 24$, then either $P_{\mathfrak{B}}(-1) = 0$ or $\lambda = 24$ and $K_{\mathfrak{B}} \sim -H_{\mathfrak{B}}$.

In the same way, one also sees that $h^4(\mathfrak{B}, \mathscr{O}_{\mathfrak{B}}) = 0$ whenever $\lambda \leq 31$. Now we discuss the cases according to the value of $a$.

*Case* B.3.1 ($9 \leq a \leq 10$). We have $\lambda \leq 8$. From the classification of del Pezzo varieties in Theorem 2.7, we see that the case $\lambda = 8$ with $K_{\mathfrak{B}} \sim -3H_{\mathfrak{B}}$ is impossible and so we obtain $\lambda = 11 - 2a/5$, $g = 1 - a/10$, by (B.3.a). Hence $a = 10$, $\lambda = 7$, $g = 0$ and $\mathfrak{B}$ is a rational normal scroll.

*Case* B.3.2 ($5 \leq a \leq 8$). We have $\lambda \leq 12$. By (B.3.b) we obtain $g = (3\lambda + a - 31)/2$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (\lambda + a - 11)/6$ and, since $\chi(\mathscr{O}_{\mathfrak{B}}) \in \mathbb{Z}$, we obtain $\lambda = 17 - a$, $g = 10 - a$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$. So, we can determine the abstract structure of $\mathfrak{B}$ by [FL97], [BB05a], [Ion86a, Theorem 2], [BB05b, Lemmas 4.1 and 6.1] and we also deduce that the case $a = 8$ does not occur, by [FL94].

*Case* B.3.3 ($a = 4$). We have $\lambda \leq 14$. Again by (B.3.b), we deduce that either $g = (3\lambda - 27)/2$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (\lambda - 7)/6$ or $\mathfrak{B}$ is a Mukai variety with $\lambda = 14$ ($g = 8$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$). In the first case, since $\chi(\mathscr{O}_{\mathfrak{B}}) \in \mathbb{Z}$ and $g \geq 0$, we obtain $\lambda = 13$, $g = 6$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$ and then we can determine the abstract structure of $\mathfrak{B}$ by [Ion85, Theorem 1] and [BB05b, Lemmas 4.1 and 6.1]. In the second case, if $b_2 = b_2(\mathfrak{B}) = 1$ then $\mathfrak{B}$ is a linear section of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$, otherwise $\mathfrak{B}$ is a Fano variety of product type, see [Muk89, Theorems 2 and 7].

*Case* B.3.4 ($a = 3$). We have $\lambda \leq 16$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 18)/3$, by (B.3.c). Moreover, if $\lambda \leq 14$, by (B.3.b) it follows that $\lambda = 14$, $g = 7$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$.

*Case* B.3.5 ($a = 2$). We have $\lambda \leq 18$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 19)/3$, by (B.3.c). Moreover, by (B.3.b) it follows that $\lambda \geq 15$.

*Case* B.3.6 ($a = 1$). We have $\lambda \leq 20$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 20)/3$, by (B.3.c). Moreover, if $\lambda \leq 14$, by (B.3.b) it follows that $\lambda = 10$, $g = 0$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 0$, which is of course impossible.

*Case* B.3.7 ($a = 0$). If $\lambda \leq 14$, by (B.3.b) and (B.3.c) it follows that $\lambda = 11$, $g = 1$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 0$. Thus, $\mathfrak{B}$ must be an elliptic scroll and $\varphi$ must be of type $(2,6)$; so, by (B.3) we obtain the contradiction $c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}}^2 = (990 + c_4(\mathfrak{B}))/37 = 990/37 \notin \mathbb{Z}$.

□

*Remark* B.4. Reasoning as in Proposition 6.9, we obtain that if $\varphi$ is of type $(2,d)$, then

$$37c_2(\mathfrak{B}) \cdot H_{\mathfrak{B}}^2 - c_4(\mathfrak{B}) = -231\lambda + 188g + (1-9d)\Delta + 3396, \tag{B.3}$$

$$37c_3(\mathfrak{B}) \cdot H_{\mathfrak{B}} + 7c_4(\mathfrak{B}) = 655\lambda - 428g + (26d-7)\Delta - 5716. \tag{B.4}$$

*Remark* B.5. If Conjecture 3.12 (with $N = 6$ and $m = 11$) holds, then we have that $\lambda \leq 24$, even in the case with $a = 0$. If $a = 0$ and $\lambda \leq 24$, we have $g \leq \theta(24) = 25$ and one of the following cases holds:

- $\lambda = 24$, $g = 25$, $\chi(\mathscr{O}_{\mathfrak{B}}) = 1$ and $\mathfrak{B}$ is a Fano variety of coindex 4;
- $g \leq 24$ and $\chi(\mathscr{O}_{\mathfrak{B}}) = (-g + 2\lambda - 21)/3$.

**Example B.6.** Below we collect some examples of special quadratic birational transformations appearing in Proposition B.3.

**($a = 10$)** If $X \subset \mathbb{P}^{10}$ is a (smooth) 4-dimensional rational normal scroll, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \mathbb{G}(1,6) \subset \mathbb{P}^{20}$ of type $(2,2)$.

($a=7$) If $X \subset \mathbb{P}^{10}$ is a general hyperplane section of $\mathbb{P}^1 \times Q^4 \subset \mathbb{P}^{11}$, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{17}$ of type $(2,2)$ whose image has degree 28.

($a=7$) If $X = \mathbb{P}(\mathscr{T}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1)) \subset \mathbb{P}^{10}$, since $h^1(X, \mathscr{O}_X) = h^1(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}) = 0$, $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{17}$ (see Facts 6.6 and 6.5).

($a=6$) There exists a smooth linearly normal 4-dimensional variety $X \subset \mathbb{P}^{10}$ with $h^1(X, \mathscr{O}_X) = 0$, degree 11, sectional genus 4, having the structure of a quadric fibration over $\mathbb{P}^1$ (see [BB05a, Remark 3.2.5]); thus $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{16}$ (see Facts 6.6 and 6.5).

($a=5$) If $X \subset \mathbb{P}^{10}$ is the blow-up of $\mathbb{P}^4$ at 4 general points $p_1, \ldots, p_4$, embedded by $|2H_{\mathbb{P}^4} - p_1 - \cdots - p_4|$, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{15}$ whose image has degree 29; in this case $\mathrm{Sec}(X)$ is a complete intersection of two cubics.

($a=4$) If $X \subset \mathbb{P}^{10}$ is a general 4-dimensional linear section of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$, then $|\mathscr{I}_{X,\mathbb{P}^{10}}(2)|$ defines a birational transformation $\psi : \mathbb{P}^{10} \dashrightarrow \overline{\psi(\mathbb{P}^{10})} \subset \mathbb{P}^{14}$ of type $(2,2)$ whose image is a complete intersection of quadrics.

*Remark* B.7. Note that in Case B.3.7 and Claim 6.19.1 we excluded the case where the base locus of a Cremona transformation is an elliptic scroll of dimension $r$ and degree $2r+3$ in $\mathbb{P}^{2r+2}$. Actually, in [ST69] is stated that the quadrics through a generic such a scroll give a Cremona transformation; although, for $r \geq 3$ the scroll is not cut out by quadrics and so the Cremona transformation is not special (see [ESB89, §4.3]).

Note also that in [HKS92, §5] a series of quadratic Cremona transformations $\psi_r : \mathbb{P}^{2r+2} \dashrightarrow \mathbb{P}^{2r+2}$ having $r$-dimensional base locus is built. For $r=2$ and $r=3$, $\psi_r$ is respectively as in Examples 6.30 and 6.37, but $\psi_4$ is not special.

## B.3 Numerical invariants of transformations of type $(2,2)$ into a quintic hypersurface

Just as in Propositions 5.30 and 5.31 (and also keeping the same notation) we can determine the possible numerical invariants for special birational transformations of type $(2,2)$ into a quintic hypersurface.

Indeed, for such a transformation, from Theorem 2.14 and Proposition 5.5, we obtain either $(n,r,\delta) = (10,4,0)$ or $\mathfrak{B}$ is a Fano variety of the first species of coindex $c(\mathfrak{B})$ and Hilbert polynomial $P$, as one of the two following cases:

(i) $n=16$, $r=8$, $\delta=2$, $c(\mathfrak{B})=4$, $P=$ 36, 216, 552, 780, 661, 340, 102, 16, 1;

(ii) $n=22$, $r=12$, $\delta=4$, $c(\mathfrak{B})=5$, $P=$ 84, 798, 3428, 8789, 14946, 17711, 14945, 9009, 3829, 1111, 207, 22, 1.

In the case in which $\mathfrak{B}$ is a 4-fold in $\mathbb{P}^{10}$, we apply Proposition B.3. So the Hilbert polynomial of $\mathfrak{B}$ can be expressed as a function of $\lambda$ and $g$. Moreover $g \equiv 1-\lambda \mod 3$ and one of the following holds: $\lambda = 15$ and $g \leq 9$; $\lambda = 16$ and $g \leq 11$; $\lambda = 17$ and $g \leq 12$; $\lambda = 18$ and $g \leq 14$; $\lambda = 19$ and $g \leq 15$; $\lambda = 20$ and $g \leq 17$.

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